2020 UI UNDERGRADUATE MATH CONTEST

1. Let $x_1, x_2, \ldots, x_{2020}$ be the roots of the polynomial $P(x) = x^{2020} + 2020x - 1$. Find, with proof, the sum
$$\sum_{i=1}^{2020} x_i^{2020}.$$

2. Without calculating the expressions involved, show that
$$2019^{2018+2020} < 2018^{2018}2020^{2020}.$$

3. Call a point (x, y) in the plane *hyperbolic* if it lies on one of the hyperbolas $y = 1/x$ or $y = -1/x$. Find a square such that its four vertices and the midpoints of its four sides are all hyperbolic.

4. Let $n \geq 3$ and let S be a family of 2^{n-1} non-empty subsets of $\{1, 2, \ldots, n\}$ such that any three sets in S have a non-empty intersection. Show that there exists an element in $\{1, 2, \ldots, n\}$ that belongs to every set in S.

5. Find all nonzero polynomials $P(x)$ with nonnegative integer coefficients such that $P(n)$ divides n^{2020} for infinitely many positive integers n.

6. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(f(n)) = n^2$ for all $n \in \mathbb{N}$? Justify your answer.

7. Let P_n be the probability that a random point (x, y), selected uniformly from the unit square $[0, 1] \times [0, 1]$, satisfies $x^n + y^n > 1$.

   (a) Show that the limit $L = \lim_{n\to\infty} n^2 P_n$ exists and satisfies $0 < L < \infty$.

   (b) Evaluate L.