2019 UI UNDERGRADUATE MATH CONTEST
Solutions

1. For any natural number n, let $f(n)$ denote the smallest natural number m such that n divides the sum $1 + 2 + \cdots + m$.

   (a) Show that $f(n) \le 2n - 1$ for all n.

   (b) For which n is $f(n) = 2n - 1$?

   Solution. (a) Since $1 + 2 + \cdots + m = m(m + 1)/2$, $f(n)$ is the smallest among the numbers m so that n divides $m(m + 1)/2$. For $m = 2n - 1$, we have $m(m + 1)/2 = (2n - 1)n$, which is always divisible by n. Thus $m = 2n - 1$ is a number that "works", and hence $f(n)$ is at most $2n - 1$.

   (b) We claim that $f(n) = 2n - 1$ if and only if n is a power of 2.

   Suppose first that $n = 2^k$ for some nonnegative integer k. Then n divides $m(m + 1)/2$ if and only if 2^{k+1} divides $m(m + 1)$, which in turn holds if and only if either m or $m + 1$ is divisible of 2^{k+1}. The smallest such m is $m = 2^{k+1} - 1 = 2n - 1$, so we have $f(n) = 2n - 1$ for $n = 2^k$.

   Now suppose that n is not a power of 2. We need to show that $f(n) < 2n - 1$.

   Write $n = 2^k a$, where k is a nonnegative integer and a is odd with $a \ge 3$. Then n divides $m(m+1)/2$ if and only if $2^{k+1}a$ divides $m(m+1)$. Since a is odd, there exists an odd integer b with $1 \le b < 2^{k+1}$ such that $ab \equiv 1 \bmod 2^{k+1}$. Let $m = ab - 1$. By the definition of b, m is divisible by 2^{k+1}, and we also have $m + 1 = ab$, which is divisible by a. Thus, $m(m + 1)/2$ is divisible by $2^k a = n$. It follows that $f(n) \le m$. Now, since $a \ge 3$ and $1 \le b < 2^{k+1}$, we have $m \ge 3b - 1 \ge 2$ and $m \le a(2^{k+1} - 1) - 1 = 2n - a - 1 < 2n - 1$. Hence $f(n) < 2n - 1$ as desired.

2. Call a set of three or more positive real numbers **balanced** if none of its elements is greater than the sum of all the other elements in the set. For example, the set $\{2, 3.1, 5\}$ is balanced, while $\{2, 3, 5.1\}$ is not balanced since $5.1 > 2 + 3$.

   Prove that any set of 13 distinct real numbers in $[1, 2019]$ contains a balanced subset.

   Solution. Let $A = \{a_1, \dots, a_{13}\}$ be a set of 13 distinct real numbers in $[1, 2019]$. Without loss of generality we may assume that $a_1 < a_2 < \cdots < a_{13}$.

   To get a contradiction, suppose A contains no balanced subset. Then, since we assumed the a_i's to be in decreasing order, we necessarily have $a_k > a_{k-1} + \cdots + a_1$ for each $k = 3, \dots, 13$. Since $a_1 \ge 1$ and $a_2 > a_1 \ge 1$, it follows that $a_3 > a_1 + a_2 > 2$, and by induction we obtain $a_k > 2^{k-2} + \dots 2^0 + 1 = 2^{k-2}$ for $k = 3, \dots, 13$, In particular, we have $a_{13} > 2^{11} > 2019$, which contradicts our assumption $A \subset [1, 2019]$. Thus A contains a balanced subset.

3. For $x > 0$, let

$$f(x) = \sum_{k=0}^{2019} \frac{1}{x^{2k/2019} + x}.$$

   Find, with proof, a simple closed formula for $f(x)$.

   Solution. We claim that $f(x) = 1010/x$. More generally, we will show that

$$f_n(x) = \sum_{k=0}^{n} \frac{1}{x^{2k/n} + x} = \frac{n+1}{2x}. \qquad (1)$$

To prove (1), let $h = n - k$, and rewrite the sum in (1) as

$$
\begin{aligned}
f_n(x) &= \sum_{h=0}^{n} \frac{1}{x^{2-2h/n} + x} = \frac{1}{x} \sum_{h=0}^{n} \frac{x^{2h/n}}{x + x^{2h/n}} \\
&= \frac{1}{x} \sum_{h=0}^{n} \left(1 - \frac{x}{x + x^{2h/n}} \right) = \frac{n+1}{x} - f_n(x).
\end{aligned}
$$

Hence $2f_n(x) = (n+1)/x$ and therefore $f_n(x) = (n+1)/(2x)$, as claimed.

4. Let
$$f(n) = \lfloor (n + \sqrt{n^2+1})^n \rfloor,$$
where $\lfloor x \rfloor$ denotes the largest integer $\le x$. Determine, with proof, all positive integers n for which $f(n)$ is even.

   Solution. We will show that $f(n)$ is even if and only if n is odd.

   For the proof write $f(n) = \lfloor \alpha_n^n \rfloor$, where $\alpha_n = n + \sqrt{n^2+1}$. Let $\beta_n = n - \sqrt{n^2+1}$. By the binomial theorem, we have

$$
\begin{aligned}
\alpha_n^n + \beta_n^n &= \sum_{k=0}^{n} \binom{n}{k} n^{n-k} \sqrt{n^2+1}^{\,k} \left(1 + (-1)^k\right) \\
&= 2 \sum_{h=0}^{\lfloor n/2 \rfloor} \binom{n}{2h} n^{n-2h} (n^2+1)^h,
\end{aligned}
$$

   which shows that $\alpha_n^n + \beta_n^n$ is always an even integer. On the other hand, since $\sqrt{n^2+1} < \sqrt{n^2+2n+1} = n+1$, we have $0 > \beta_n > -1$. Thus β_n^n is a number in $(0,1)$ if n is even and in $(-1,0)$ if n is odd. It follows that α_n^n falls into an interval of the form $(2k-1, 2k)$ when n is even and $(2k, 2k+1)$ when n is odd. In the first case, $f(n) = \lfloor \alpha_n^n \rfloor$ is odd, while in the second case $f(n)$ is even. Hence $f(n)$ is even if and only if n is odd, as claimed.

5. Let $A = \{4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, \dots\}$ be the set of all distinct integers of the form n^k, where n and k are integers with $n \ge 2$ and $k \ge 2$. (Note that integers such as $16 = 2^4 = 4^2$ that have multiple representations as powers are counted only once in A.)

   Evaluate, with proof, the infinite series
$$\sum_{a \in A} \frac{1}{a-1}.$$

   Solution. We will show that the sum is equal to 1.

   Let B be the set of natural numbers ≥ 2 that are not in A. Then each element of A can be written uniquely as $a = b^k$, where $k \geq 2$ and $b \in B$. Now,

$$\begin{aligned}
\sum_{a\in A} \frac{1}{a-1} &= \sum_{b\in B}\sum_{k=2}^{\infty} \frac{1}{b^k-1} \\
&= \sum_{b\in B}\sum_{k=2}^{\infty}\sum_{h=1}^{\infty} \frac{1}{b^{kh}} \\
&= \sum_{b\in B}\sum_{h=1}^{\infty}\sum_{k=2}^{\infty} \frac{1}{b^{kh}} \\
&= \sum_{b\in B}\sum_{h=1}^{\infty} \frac{1}{b^{2h}-b^h} \\
&= \sum_{b\in B}\sum_{h=1}^{\infty} \left(\frac{1}{b^h-1} - \frac{1}{b^h}\right).
\end{aligned}$$

Now, by the definition of B, the numbers b^h are exactly the positive integers $n \geq 2$, with each n occurring exactly once as $n = b^h$. Thus, the latter sum is equal to the telescoping series

$$\sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1.$$

6. A set of positive integers is called **progression-free** if it does not contain an arithmetic progression of length at least 3. For example, the set $\{5, 7, 11, 13, 17\}$ is not progression-free since $5, 11, 17$ form an arithmetic progression, but $\{7, 11, 13, 17\}$ is progression-free.

   Show that the set $\{0, 1, \ldots, 3^{2019} - 1\}$ contains a progression-free subset with 2^{2019} elements.

   Solution. More generally, we will show that for any positive integer n, there exists a progression-free set of 2^n integers in $[0, 3^n - 1]$.

   Let n be given and define S as the set of all integers in $[0, 3^n - 1]$ whose ternary (i.e., base 3) expansion contains only 0's and 1's. By padding the expansion with 0's at the beginning if needed, we can write the elements of S in the form $(a_{n-1}a_{n-2}\ldots a_0)_3$ with $a_i \in \{0, 1\}$. Thus, S has exactly 2^n elements.

   We now show that S is progression-free. It suffices to show that S has no 3-term arithmetic progression. To get a contradiction, suppose that a, b, c are elements in S with $a < b < c$ that form a 3-term arithmetic progression. Write the ternary expansions of a, b, c as $a = (a_{n-1}\ldots a_0)_3$, $b = (b_{n-1}\ldots b_0)_3$, and $c = (c_{n-1}\ldots c_0)_3$. By our assumption that a, b, c form a 3-term arithmetic progression, we have $2b = a + c$. Since the digits a_i, b_i, and c_i are in $\{0, 1\}$, the numbers $2b$ and $a + c$ have ternary expansions $2b = ((2b_{n-1})\ldots(2b_0))_3$ and $a + c = ((a_{n-1} + c_{n-1})\ldots(a_0 + c_0))_3$, respectively. It follows that $2b_i = a_i + c_i$ for each i, and in particular that $a_i + c_i$ is even for each i. But since $a_i, c_i \in \{0, 1\}$, this is only possible if $a_i = c_i$ for each i, and hence $a = c$. This contradicts our assumption $a < c$.

7. Let M be the set of all 40×40 matrices with elements ± 1. Let R_i denote the operation that switches all signs in row i (i.e., replaces all $+1$'s by -1's and all -1's by $+1$'s in this row); similarly, let C_j denote the operation that switches all signs in column j. Thus, each such operation transforms a matrix in M to another matrix in M.

   If you start out with the matrix in M in which all 1600 entries are $+1$, is there a finite sequence of such operations that transforms this matrix into one consisting of exactly 180 entries -1 and 1420 elements $+1$? Explain your answer.

Solution. We will show that this is not possible.

Observe first that the transformed matrix has a minus sign in an entry if and only if either the corresponding row or the corresponding column (but not both) was flipped an odd number of times. It follows that the total number of minus signs in the transformed matrix is equal to $40(a+b)-2ab$, where a and b denote the number of columns and rows that have been flipped an odd number of times,

We need to show that this expression can never be equal to 180. Indeed, suppose $40(a+b)-2ab = 180$. Then

$$\begin{aligned} a(40-b)+b(40-a) &= 180, \\ 20a+20b-ab &= 90, \\ (a-20)(b-20) &= 310 = 31\cdot 10. \end{aligned}$$

The number 31 is prime, hence it divides either $a-20$ or $b-20$. But this is impossible since a and b are both integers in $\{0,1,\dots,40\}$.