2019 UI UNDERGRADUATE MATH CONTEST

1. For any natural number n, let $f(n)$ denote the smallest natural number m such that n divides the sum $1 + 2 + \cdots + m$.

   (a) Show that $f(n) \le 2n - 1$ for all n.

   (b) For which n is $f(n) = 2n - 1$?

2. Call a set of three or more positive real numbers **balanced** if none of its elements is greater than the sum of all the other elements in the set. For example, the set $\{2, 3.1, 5\}$ is balanced, while $\{2, 3, 5.1\}$ is not balanced since $5.1 > 2 + 3$.

   Prove that any set of 13 distinct real numbers in $[1, 2019]$ contains a balanced subset.

3. For $x > 0$, let
$$f(x) = \sum_{k=0}^{2019} \frac{1}{x^{2k/2019} + x}.$$
   Find, with proof, a simple closed formula for $f(x)$.

4. Let
$$f(n) = \lfloor (n + \sqrt{n^2 + 1})^n \rfloor,$$
   where $\lfloor x \rfloor$ denotes the largest integer $\le x$. Determine, with proof, all positive integers n for which $f(n)$ is even.

5. Let $A = \{4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, \dots\}$ be the set of all distinct integers of the form n^k, where n and k are integers with $n \ge 2$ and $k \ge 2$. (Note that integers such as $16 = 2^4 = 4^2$ that have multiple representations as powers are counted only once in A.)

   Evaluate, with proof, the infinite series
$$\sum_{a \in A} \frac{1}{a - 1}.$$

6. A set of positive integers is called **progression-free** if it does not contain an arithmetic progression of length at least 3. For example, the set $\{5, 7, 11, 13, 17\}$ is not progression-free since $5, 11, 17$ form an arithmetic progression, but $\{7, 11, 13, 17\}$ is progression-free.

   Show that the set $\{0, 1, \dots, 3^{2019} - 1\}$ contains a progression-free subset with 2^{2019} elements.

7. Let M be the set of all 40×40 matrices with elements ± 1. Let R_i denote the operation that switches all signs in row i (i.e., replaces all $+1$'s by -1's and all -1's by $+1$'s in this row); similarly, let C_j denote the operation that switches all signs in column j. Thus, each such operation transforms a matrix in M to another matrix in M.

   If you start out with the matrix in M in which all 1600 entries are $+1$, is there a finite sequence of such operations that transforms this matrix into one consisting of exactly 180 entries -1 and 1420 elements $+1$? Explain your answer.