2018 UI UNDERGRADUATE MATH CONTEST SOLUTIONS

1. Find a polynomial $P(x)$ with integer coefficients such that

$$P(2017) = 2018,\ P(2018) = 2019,\ P(2019) = 2020,\ P(2020) = 2017,$$

or prove that there is no such polynomial.

Solution. We will show that there is no such polynomial. Suppose $P(x)$ is a polynomial with the desired properties. Then, since $2020 \equiv 2017 \bmod 3$, we have $P(2020) \equiv P(2017) \bmod 3$, so 3 must divide $P(2020) - P(2017)$. But, by the given values, we have $P(2020) - P(2017) = 2017 - 2018 = -1$, so we have a contradiction. (This argument uses the following general property of congruences: If $a \equiv b \bmod m$, then for any polynomial $P(x)$ with integer coefficients, $P(a) \equiv P(b) \bmod m$.)

2. Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be $2n$ positive real numbers. Show that at least one of the inequalities

$$\frac{a_1}{b_1} + \dots + \frac{a_n}{b_n} \geq n \tag{1}$$

or

$$\frac{b_1}{a_1} + \dots + \frac{b_n}{a_n} \geq n \tag{2}$$

holds.

Solution. By the Cauchy-Schwarz inequality, we have

$$n^2 = \left(\sum_{i=1}^{n} \sqrt{a_i/b_i} \cdot \sqrt{b_i/a_i} \right)^2 \leq \left(\sum_{i=1}^{n} \frac{a_i}{b_i} \right) \left(\sum_{i=1}^{n} \frac{b_i}{a_i} \right) \tag{3}$$

If the inequalites (1) and (2) were both false, then the product on the right of (3) would be $< n^2$, contradicting (3). Thus, at least one of (1) and (2) must hold.

3. Call a permutation of $1, 2, \dots, n$ **good** if each element either stays in its place, or moves left or right by one spot. For example, the permutation $1, 3, 2, 5, 4$ is a good permutation of $1, 2, \dots, 5$, generated by the moves $1 \to 1$, $2 \to 3$, $3 \to 2$, $4 \to 5$, $5 \to 4$.

Let a_n denote the number of good permutations of $1, 2, \dots, n$. Find, with proof, a general formula for a_n and express your answer in terms of a famous sequence.
(A permutation of $1, 2, \dots, n$ is an arrangement of these numbers in some order.)

Solution. We claim that $(*)$ $a_n = F_{n+1}$, where F_n is the n-th Fibonacci number. (The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 2$.)
To prove $(*)$, we use strong induction. For $n = 1$, there is only one permutation, namely the identity permutation, which is obviously good. Thus, $a_1 = 1 = F_2$. For $n = 2$, there are two permutations, $1, 2$ and $2, 1$, both of which are good, so we have $a_2 = 2 = F_3$. Thus, $(*)$ holds for $n = 1$ and $n = 2$.

Now let $k \geq 2$ be given and suppose $(*)$ holds for $n = k - 1$ and $n = k$. We seek to show that $(*)$ holds for $n = k + 1$. The good permutations of $1, 2, \ldots, k + 1$ are of one of two types: (I) permutations that leave the element $k + 1$ fixed; and (II) permutations that interchange $k + 1$ with k.

The good permutations of type (I) are in one-to-one correspondence with good permutations of $1, 2, \ldots, k$, and by the induction hypothesis there are $a_k = F_{k+1}$ such permutations. Similarly, the good permutations of type (II) are in one-to-one correspondence with good permutations of $1, 2, \ldots, k - 1$, and by the induction hypothesis there are $a_{k-1} = F_k$ such permutations. Hence the total number of good permutations of $1, 2, \ldots, k + 1$ is $a_{k+1} = a_k + a_{k-1} = F_{k+1} + F_k = F_{k+2}$. This proves $(*)$ for $n = k + 1$ and completes the induction.

4. Determine, with proof, all positive real numbers p for which the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n \sin(1/n)} - 1 \right)^p$$

converges.

Solution. We claim that the series converges if and only if $p > 1/2$.

For the proof, let a_n be the expression in parentheses, and write

$$a_n = \frac{1 - n \sin(1/n)}{n \sin(1/n)} = \frac{r_n}{q_n},$$

say. Using the Taylor expansion $\sin(1/n) = (1/n) - (1/n)^3/3! + O(1/n^5)$, we have

$$q_n = n \sin(1/n) = n \left(\frac{1}{n} + O\left(\frac{1}{n^3} \right) \right) = 1 + O\left(\frac{1}{n^2} \right),$$
$$r_n = 1 - n \left(\frac{1}{n} - \frac{1}{6n^3} + O\left(\frac{1}{n^5} \right) \right) = \frac{1}{n^2} \left(1 + O\left(\frac{1}{n^2} \right) \right).$$

Thus $a_n n^2 = r_n n^2 / q_n \to 1/6$ as $n \to \infty$. It follows that for sufficiently large n, we have $(1/10) n^{-2} \leq a_n \leq n^{-2}$. Therefore the series $\sum_{n=1}^{\infty} a_n^p$ converges if and only if the series $\sum_{n=1}^{\infty} n^{-2p}$ converges. But the latter series converges if and only if $p > 1/2$. This proves our claim.

5. Given n random points on a circle, find, with proof, the probability that the convex polygon formed by these points does *not* contain the center of the circle.

Solution. We claim that the desired probability is $n 2^{-n+1}$.

For the proof, let $P_1, P_2, \ldots, P_n$ be the n random points on the circle. Call a point P_i *special* if the other $(n - 1)$ points P_j all lie on the open semi-circle starting at P_i and proceeding in *clockwise* direction. Clearly, if P_i is special, then the convex polygon formed by the points $P_1, \ldots, P_n$ does not contain the center of the circle; conversely, if this polygon does not contain the center of the circle, exactly one of the n points P_i must be special. The probability asked for is therefore the probability that *some* point P_i is special.

The probability that a *given* point P_i is special is just the probability that each of the other $n - 1$ points P_j falls into a given semi-circle, i.e., $(1/2)^{n-1}$. Since at most one point P_i can be special, the events "P_i is special", $i = 1, \ldots, n$, are pairwise disjoint, so their union, i.e., the event "some point P_i is special", has probability $n(1/2)^{n-1}$, as claimed.

6. Call a set A of positive integers **divisorful** if for any pair of distinct elements in A one of these two elements divides the other, and call A **divisorfree** if there exists *no* pair of distinct elements in A such that one of these two elements divides the other. For example, the set $\{2, 6, 18, 36\}$ is divisorful, while the set $\{6, 8, 10, 15\}$ is divisorfree.

Let k and n be arbitrary positive integers.

(a) Find a set of kn distinct positive integers that does not contain a divisorful subset with more than k elements or a divisorfree subset with more than n elements.

(b) Show that any set of $kn + 1$ distinct positive integers contains either a divisorful subset with more than k elements or a divisorfree subset with more than n elements (or both).

Solution. (a) Let $p_1, \dots, p_k$ denote the first k primes, and set $a_{ij} = p_i^j$ for $i = 1, 2, \dots, k$, $j = 1, 2, \dots, n$. We claim that the set S of these numbers a_{ij} has the desired properties.

For the proof, first note that the numbers a_{ij} are pairwise distinct, so S has kn elements. Moreover, if $i \neq i'$, then, for any $j, j' \leq n$, the numbers a_{ij} and $a_{i'j'}$ involve two distinct prime factors, p_i and $p_{i'}$, so neither of these numbers is a divisor of the other. On the other hand, if $i = i'$, then the numbers a_{ij} and $a_{i'j'}(= a_{ij'})$ differ by a factor $p_i^{j-j'}$, so one of these numbers divides the other.

Now suppose A is a subset of S with more than k elements. By the pigeonhole principle, A must contain two elements with the same first index i, and by the above remark one of these two elements must divide the other. Therefore A cannot be divisorfree.

Similarly, if A is a subset with more than n elements, then by the pigeonhole principle A must contain two distinct elements with the same second index j. Since these elements are distinct, they must have distinct first indices, and by the above remark we see that neither of these elements divides the other. Therefore A cannot be divisorful.

Thus, the set S of elements a_{ij} of the above form is a set of kn elements with the required properties.

(b) Let S be a given set of $kn + 1$ distinct positive integers. For each $s \in S$, let A_s be a maximal divisorful subset of S containing s. Writing the elements of A_s in increasing order as $s_1 < s_2 < \cdots < s_h$, the "divisorful" property of A_s implies that $s_1 | s_2 | \cdots | s_h$ (where $d|e$ means that d divides e). Since s is an element of A_s, it must be equal to s_r for some r. Call r the *rank* of s, and denote it by $r(s)$. Note that $r(s)$ does not depend on the choice of the set A_s. Indeed, suppose there is another maximal divisorful set A'_s in which s has a larger (or smaller) rank. Then by combining the elements $\leq s$ in one of the two sets with the elements $> s$ in the other set one can construct a divisorful set with larger cardinality than A_s and A'_s, contradicting the maximality of these sets.

If one of the sets A_s has more than n elements, then this set is a divisorful subset of S with more than n elements and we are done. Suppose therefore that each of the sets A_s has at most n elements. Then the rank, $r(s)$, can have at most n distinct values. Since S has $nk + 1$ elements, the pigeonhole principle implies that there exists an r such that $r = r(s)$ for at least $k + 1$ elements s. Let A denote the set of these $k + 1$ elements. We claim that A is divisorfree. Indeed, if s and s' are two distinct elements in A such that s divides s', then combining the elements of A_s that are $\leq s$ with those of $A_{s'}$ that are $\geq s'$ would give a divisorful set containing s' that has one more element than $A_{s'}$, contradicting the maximality of $A_{s'}$. Thus the set A constructed in this way is a divisorfree set with at least $k + 1$ elements, and the proof is complete.