2018 UI UNDERGRADUATE MATH CONTEST

1. Find a polynomial $P(x)$ with integer coefficients such that

$$P(2017) = 2018, \ P(2018) = 2019, \ P(2019) = 2020, \ P(2020) = 2017,$$

   or prove that there is no such polynomial.

2. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ positive real numbers. Show that at least one of the inequalities

   (1)
   $$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \geq n$$

   or

   (2)
   $$\frac{b_1}{a_1} + \cdots + \frac{b_n}{a_n} \geq n$$

   holds.

3. Call a permutation of $1, 2, \ldots, n$ **good** if each element either stays in its place, or moves left or right by one spot. For example, the permutation $1, 3, 2, 5, 4$ is a good permutation of $1, 2, \ldots, 5$, generated by the moves $1 \to 1$, $2 \to 3$, $3 \to 2$, $4 \to 5$, $5 \to 4$.

   Let a_n denote the number of good permutations of $1, 2, \ldots, n$. Find, with proof, a general formula for a_n and express your answer in terms of a famous sequence.
   (A permutation of $1, 2, \ldots, n$ is an arrangement of these numbers in some order.)

4. Determine, with proof, all positive real numbers p for which the series

   $$\sum_{n=1}^{\infty} \left(\frac{1}{n \sin(1/n)} - 1 \right)^p$$

   converges.

5. Given n random points on a circle, find, with proof, the probability that the convex polygon formed by these points does *not* contain the center of the circle.

6. Call a set A of positive integers **divisorful** if for any pair of distinct elements in A one of these two elements divides the other, and call A **divisorfree** if there exists *no* pair of distinct elements in A such that one of these two elements divides the other. For example, the set $\{2, 6, 18, 36\}$ is divisorful, while the set $\{6, 8, 10, 15\}$ is divisorfree.

   Let k and n be arbitrary positive integers.

   (a) Find a set of kn distinct positive integers that does not contain a divisorful subset with more than k elements or a divisorfree subset with more than n elements.

   (b) Show that any set of $kn + 1$ distinct positive integers contains either a divisorful subset with more than k elements or a divisorfree subset with more than n elements (or both).