2017 UI UNDERGRADUATE MATH CONTEST SOLUTIONS

1. Let $a_1 = 1, a_2 = 2, \ldots, a_{2017} = 2017$, and for $n \geq 2017$ let

$$a_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n} a_k.$$

Find, with proof, $\lim_{n\to\infty} a_n$.

Solution. We claim that $a_n = 2017/2$ for $n \geq 2018$, so the above limit exists and is equal to $\boxed{2017/2}$.
For the proof, set $N = 2017$, so that we need to prove $a_{N+n} = N/2$ for $n = 1, 2, \ldots$.
Let S be the sum of the first $N(= 2017)$ terms of the sequence, so that $S = \sum_{k=1}^{N} k = N(N+1)/2$. Then, for $n = 1, 2, \ldots,$

$$a_{N+n} = \frac{1}{N+n}\left(S + \sum_{k=1}^{n-1} a_{N+k}\right) = \frac{1}{N+n}\left(\frac{N(N+1)}{2} + \sum_{k=1}^{n-1} a_{N+k}\right).$$

We will show by induction that $(*)$ $a_{N+n} = N/2$ holds for each $n = 1, 2, \ldots$.
Setting $n = 1$ gives $a_{N+1} = S/(N+1) = N/2$, so $(*)$ holds in the base case, $n = 1$. Now suppose $(*)$ holds for some $n \geq 1$. Then

$$a_{N+n+1} = \frac{1}{N+n+1}\left(\frac{N(N+1)}{2} + \sum_{k=1}^{n} \frac{N}{2}\right) = \frac{N}{2},$$

completing the induction. This proves $(*)$, and the above claim.
Remark: This problem is based on a problem from the 1986 Virginia Tech Math Contest.

2. Evaluate the sum

$$S(n) = \binom{n}{n}2^n + \binom{n+1}{n}2^{n-1} + \cdots + \binom{2n}{n}2^0.$$

Solution. Working out the first few cases suggests the answer $(*)$ $\boxed{S(n) = 2^{2n}}$.
To prove $(*)$, it suffices to show that (1) $S(1) = 2^2 = 4$ and (2) $S(n+1) = 4S(n)$ for $n \geq 1$. For $n = 1$, we have $S(1) = \binom{1}{1} \cdot 2^1 + \binom{2}{1} \cdot 2^0 = 2^2$, proving (1).
Using the identity

$$\binom{n+1+h}{n+1} = \binom{n+h}{n} + \binom{n+h}{n+1},$$

we get

$$\begin{aligned}
S(n+1) &= \sum_{h=0}^{n+1} \binom{n+1+h}{n+1} 2^{n+1-h} \\
&= \sum_{h=0}^{n+1} \binom{n+h}{n} 2^{n+1-h} + \sum_{h=1}^{n+1} \binom{n+h}{n+1} 2^{n+1-h} \\
&= \sum_{h=0}^{n} \binom{n+h}{n} 2^{n+1-h} + \binom{2n+1}{n+1} 2^0 \\
&\quad + \sum_{k=0}^{n+1} \binom{n+1+k}{n+1} 2^{n+1-k-1} - \binom{2n+2}{n+1} 2^{-1} \\
&= 2S(n) + \frac{1}{2}S(n+1) + \left(\binom{2n+1}{n+1} - \frac{1}{2}\binom{2n+2}{n+1}\right) \\
&= 2S(n) + \frac{1}{2}S(n+1),
\end{aligned}$$

since

$$\binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n} = 2\binom{2n+1}{n+1}.$$

Hence $S(n+1) = 4S(n)$, proving (2).

Alternate proof: A combinatorial proof of $S(n) = 2^{2n}$ goes as follows: On the one hand, 2^{2n} is the number of 01-sequences of length $2n$. On the other hand, we can count these sequences by considering the minimal index, $n+h$, such that among the first $n+h$ terms there are exactly n 1's (Case I) or exactly n 0's (Case II). For a given value of h the number of such sequences turns out to be $2\binom{n+h-1}{n-1} \cdot 2^{n-h}$, and summing over h one gets the sum $S(n)$, after some manipulation.

3. Let the sequence $\{a_n\}$ be defined by $a_0 = 0$, $a_1 = 1$, and

$$a_{2n} = a_n, \quad a_{2n+1} = a_n + 1 \quad (n = 1, 2, \dots).$$

Evaluate the infinite series

$$\sum_{n=1}^{\infty} \frac{a_n}{n(n+1)}.$$

Solution. Let S denote the sum of the given series. We will show that $\boxed{S = \log 4}$.

Splitting the series into odd and even parts, we obtain

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{a_{2n}}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{a_{2n+1}}{(2n+1)(2n+2)} \\ &= \sum_{n=1}^{\infty} \frac{a_n}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{a_n + 1}{(2n+1)(2n+2)} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} + \sum_{n=1}^{\infty} \left(\frac{a_n}{(2n)(2n+1)} + \frac{a_n}{(2n+1)(2n+2)} \right) \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n}{n(n+1)} = \log 2 + \frac{1}{2} S, \end{aligned}$$

where we have used the logarithmic series $\log(1+x) = -\sum_{n=1}^{\infty} (-1)^n x^n / n$, which is valid for $x = 1$. Hence, $S = \log 2 + S/2$, and so $S = 2 \log 2 = \log 4$, as claimed.

Remarks: This problem is a variation of Problem B5 from the 1981 Putnam exam. The Putnam version involved a different, but equivalent, definition of a_n, namely as the number of 1's in the binary expansion of n.

4. Find, with proof, all integers $n \geq 3$ for which there is a polynomial of degree n with the following properties:

   (a) $P(k) = k^3$ for $k = 1, 2, \dots, n$;

   (b) $P(0)$ is an integer;

   (c) $P(-1) = 2017$.

Solution. We will show that the n-values with the given properties are $\boxed{n = 2017, 1008}$.

For the proof, suppose first that $P(x)$ is a polynomial of degree $n \geq 3$ satisfying the three conditions (a), (b), and (c). Consider the polynomial $Q(x) = P(x) - x^3$. Then $Q(x)$ has degree at most n, and condition (a) implies that $Q(x)$ has a root at each of the numbers $k = 1, 2, \dots, n$. It follows that $Q(x)$ is of the form $Q(x) = C(x-1)(x-2)\dots(x-n)$ for some constant C. To determine C, we use condition (c), which implies

$$2017 = P(-1) = Q(-1) + (-1)^3 = C(-1)^n (n+1)! - 1.$$

Hence $C = 2018(-1)^n/(n+1)!$. It follows that

$$P(0) = Q(0) = C(-1)^n n! = \frac{2018}{n+1},$$

so condition (b) holds if and only if $n+1$ is a divisor of 2018. Now, $2018 = 2 \cdot 1009$, where 1009 is prime, so the divisors of 2018 are $2018, 1009, 2, 1$, with corresponding n-values of $2017, 1008, 1, 0$. The condition $n \geq 3$ eliminates the last two values, so we necessarily have $n \in \{2017, 1008\}$.

Conversely, if $n \in \{2017, 1008\}$, then $2018/(n+1)$ is an integer, so defining $P(x)$ as above (i.e., $P(x) = x^3 + C(x-1)(x-2)\dots(-n)$, with $C = 2018(-1)^n/(n+1)!$), all three conditions (a)–(c) are satisfied. Therefore the numbers n sought in the problem are exactly $n = 2017$ and $n = 1008$, as claimed.

5. Prove that the product of three consecutive positive integers is never a perfect power. (A perfect power is an integer of the form m^k, where m, k are both integers ≥ 2.)

Solution. Writing the three consecutive integers as $n-1, n, n+1$, the problem amounts to showing that the equation

$$(*) \qquad (n-1)n(n+1) = m^k$$

has no solution in integers $n, m, k \geq 2$.

We argue by contradiction. Suppose there exist $n, m, k \geq 2$ such that $(*)$ holds. Consider the prime factorization of m. Note that $(n-1)n(n+1) = n(n^2-1)$, where the two factors on the right, n and n^2-1, are relatively prime. Therefore, any prime factor of m must be either a prime factor of n or a prime factor of n^2-1, but not both. Thus, we can split m as $m = ab$, where a contains the prime factors of m common with n, and b contains the prime factors of m common with n^2-1. Then $n(n^2-1) = a^k b^k$, and since $(a, n^2-1) = 1$ and $(b, n) = 1$, we must have $n = a^k$ and $n^2 - 1 = b^k$. It follows that

$$(1) \qquad b^k = n^2 - 1 = (a^k)^2 - 1 = (a^2)^k - 1.$$

Therefore $b < a^2$, and since a and b are integers, we must have $b \leq a^2 - 1$, and hence

$$b^k \leq (a^2-1)^k < (a^2)^k - 1.$$

But this contradicts (1), proving our claim.

Remark: This problem is #2 on the 1984 Virginia Tech Contest.

6. Evaluate the n-dimensional integral

$$I_n = \int \cdots \int_{R_n} \cos^2(x_1)\cos^2(x_1+x_2)\cdots\cos^2(x_1+x_2+\cdots+x_n)\,dV,$$

where R_n is the set of points $(x_1, \ldots, x_n)$ in n-dimensional space satisfying

$$0 \leq x_i \leq \pi \quad (i = 1, 2, \ldots, n), \quad x_1 + x_2 + \cdots + x_n \leq \pi.$$

Solution. We will show that the integral is equal to $I_n = \boxed{\pi^n/(n!2^n)}$

The form of the integrand function and the region R_n suggests making the linear change of variables

$$(1) \qquad y_k = x_1 + \cdots + x_k, \quad k = 1, 2, \ldots, n.$$

In matrix form, this can be written as $\mathbf{y} = A\mathbf{x}$, where $\mathbf{y}$ and $\mathbf{x}$ are column vectors with components y_i and x_i, respectively, and A is the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0\ldots \\ 1 & 1 & 0 & 0\ldots \\ 1 & 1 & 1 & 0\ldots \\ & \ldots & \ldots & \ldots \end{bmatrix}$$

Since A has determinant 1, the Jacobian determinant corresponding to the change of variables (1) is also 1. Rewriting the given integral in terms of the variables y_i we get

$$(2) \qquad I_n = \int \cdots \int_{S_n} \cos^2(y_1)\cos^2(y_2)\cdots\cos^2(y_n)dV,$$

where S_n is the $(y_1, \ldots, y_n)$-region corresponding to R_n, namely

$$0 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq \pi.$$

The integral (2) can be evaluated by a symmetry argument: Note that the region S_n represents the part of the n-dimensional cube $[0, \pi]^n$ in which variables $y_1, \ldots, y_n$ are in a particular order (namely increasing). Since the integrand is symmetric in the variables $y_1, \ldots, y_n$, integrating over any other particular ordering of these variables inside the cube $[0, \pi]^n$ produces the same value. Since there are $n!$ such orderings, the integral over any given ordering is $1/n!$ times the integral over the full n-dimensional cube. Therefore,

$$\begin{aligned} I_n &= \frac{1}{n!}\int_{y_n=0}^{\pi}\int_{y_{n-1}=0}^{\pi}\cdots\int_{y_1=0}^{\pi}\cos^2(y_1)\ldots\cos^2(y_{n-1})\cos^2(y_n)dy_1\ldots dy_{n-1}dy_n \\ &= \frac{1}{n!}\left(\int_0^{\pi}\cos^2 y dy\right)^n = \frac{1}{n!}(\pi/2)^n \end{aligned}$$

as claimed.