2017 UI UNDERGRADUATE MATH CONTEST

1. Let $a_1 = 1, a_2 = 2, \ldots, a_{2017} = 2017$, and for $n \geq 2017$ let

$$a_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n} a_k.$$

Find, with proof, $\lim_{n\to\infty} a_n$.

2. Evaluate the sum

$$S(n) = \binom{n}{n} 2^n + \binom{n+1}{n} 2^{n-1} + \cdots + \binom{2n}{n} 2^0.$$

3. Let the sequence $\{a_n\}$ be defined by $a_0 = 0$, $a_1 = 1$, and

$$a_{2n} = a_n, \quad a_{2n+1} = a_n + 1 \quad (n = 1, 2, \ldots).$$

Evaluate the infinite series

$$\sum_{n=1}^{\infty} \frac{a_n}{n(n+1)}.$$

4. Find, with proof, all integers $n \geq 3$ for which there is a polynomial of degree n with the following properties:

   (a) $P(k) = k^3$ for $k = 1, 2, \ldots, n$;
   (b) $P(0)$ is an integer;
   (c) $P(-1) = 2017$.

5. Prove that the product of three consecutive positive integers is never a perfect power. (A perfect power is an integer of the form m^k, where m, k are both integers ≥ 2.)

6. Evaluate the n-dimensional integral

$$I_n = \int \cdots \int_{R_n} \cos^2(x_1)\cos^2(x_1 + x_2) \cdots \cos^2(x_1 + x_2 + \cdots + x_n)\, dV,$$

where R_n is the set of points $(x_1, \ldots, x_n)$ in n-dimensional space satisfying

$$0 \leq x_i \leq \pi \quad (i = 1, 2, \ldots, n), \quad x_1 + x_2 + \cdots + x_n \leq \pi.$$