1. Given positive integers \( n \) and \( k \), let \( f_k(n) \) be the number of ordered \( k \)-tuples \((a_1, a_2, \ldots, a_k)\) of positive integers such that \( n = a_1 \cdot a_2 \cdots a_k \). For example, \( f_2(10) = 4 \) since there are 4 pairs \((a_1, a_2)\) of positive integers with product 10: \((1, 10), (2, 5), (5, 2), (10, 1)\).

(a) Find \( f_2(2016) \). (Note that 2016 = \( 2^5 \cdot 3^2 \cdot 7^1 \).)

(b) Find, with proof, a simple formula for \( f_k(2015) \), where \( k \) is an arbitrary positive integer. (Note that 2015 = \( 5 \cdot 13 \cdot 31 \).)

**Solution.** (a) Since 2016 = \( 2^5 \cdot 3^2 \cdot 7 \), we have 2016 = \( a_1 a_2 \) if and only if \( a_1 \) is of the form (*) \( a_1 = 2^\alpha 3^\beta 7^\gamma \), where \( \alpha \in \{0, 1, \ldots, 5\} \), \( \beta \in \{0, 1, 2\} \), and \( \gamma \in \{0, 1\} \). There are 6 choices for \( \alpha \), 3 for \( \beta \), and 2 for \( \gamma \), so the total number of integers \( a_1 \) of the form (*) is \( 6 \cdot 3 \cdot 2 = 36 \).

(b) Since 2015 = \( 5 \cdot 13 \cdot 31 \), we have 2015 = \( a_1 a_2 \cdots a_k \) if and only if each of the three prime factors 5, 13, 31 is a prime factor of exactly one \( a_i \), and the \( a_i \)'s do not contain any prime factors other than those three. Since there are \( k^3 \) ways to distribute the three prime factors among the \( a_i \)'s, we have \( f_k(2015) = k^3 \).

2. Given two positive integers \( n \) and \( m \), call \( m \) a **descendant** of \( n \) if \( m \) can be obtained from \( n \) by replacing zero or more of its non-zero digits (in decimal representation) by 0. For example, the number 213 has 7 non-zero descendants: 213, 210, 203, 013(=13), 200, 010(=10), 003(=3).

Prove that any positive integer containing exactly 2016 digits in its decimal representation and none of whose digits is zero has a non-zero descendant that is divisible by 2016.

**Solution.** Let \( N \) be the given integer, and consider the integers \( N_k \), \( k = 0, 1, \ldots, 2015 \), obtained from \( N \) by replacing the last \( k \) digits by 0's. (In particular, \( N_0 = N \) is the given integer.) Since \( N \) has 2016 digits, none of which is 0, the integers \( N_k \) are all distinct, non-zero descendants of \( N_0 \).

If one of the integers \( N_0, N_1, \ldots, N_{2015} \) is divisible by 2016, we have obtained the desired conclusion.

Now consider the remaining case, i.e., the case when none of the integers \( N_k \) is congruent to 0 modulo 2016. Since there are 2016 such integers and 2015 non-zero congruence classes modulo 2016, by the pigeonhole principle two of these integers, say \( N_h \) and \( N_k \) with \( h < k \), must fall into the same congruence class modulo 2016. Then \( N_h - N_k \) is divisible by 2016. But \( N_h - N_k \) is the integer obtained from \( N \) by replacing the last \( h \) and the first \( 2016 - k \) digits by 0's (with the remaining \( k - h \) digits being non-zero), and hence is a non-zero descendant of \( N \). Thus the desired conclusion holds in this case as well.

3. Let \( C_1 \) be the unit circle \( x^2 + y^2 = 1 \), and let \( C_r \) denote the circle \( x^2 + y^2 = r^2 \), where \( r \) is a given real number with \( 0 < r < 1 \). Two points \( P \) and \( Q \) are chosen randomly and independently on the circumference of \( C_1 \). Find, with proof, the probability that the line segment \( PQ \) intersects the circle \( C_r \).

**Solution.** Let \( O \) denote the center of the two circles, \( M \) the midpoint of the line segment \( PQ \), and \( \alpha \) the angle at \( O \) of the triangle \( POQ \). Then \( PQ \) intersects the circle \( C_r \) if and only if \( r \geq |OM| \). Now, \( |OM| = \cos(\alpha/2) \), so the latter condition is equivalent to \( r \geq \cos(\alpha/2) \), or \( \alpha \geq 2 \arccos(r) \).
Now the angle $\alpha$ represents the length of the shorter of the arcs between $P$ and $Q$ on the unit circle, and since $Q$ is a random point on this circle, the angle $\alpha$ is uniformly distributed between $0$ and $\pi$. Hence

$$P(\alpha \geq 2 \arccos(r)) = \frac{1}{\pi}(\pi - 2 \arccos(r)) = \frac{2}{\pi}\arcsin(r)$$

4. Given a real number $x$ such that $x > 1$, define a sequence $a_1, a_2, a_3, \ldots$ by $a_1 = x$, and

$$a_{n+1} = a_n^2 - a_n + 1 \quad (n = 1, 2, 3, \ldots).$$

Show that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and find its value, as a function of $x$.

**Solution.** We claim that the series converges, with sum $1/(x - 1)$.

Writing the given recurrence as $a_{n+1} - 1 = a_n(a_n - 1)$, we get

$$\frac{1}{a_{n+1} - 1} = \frac{1}{a_n(a_n - 1)} = \frac{1}{a_n - 1} - \frac{1}{a_n}.$$

Hence, for any positive integer $N$ we have

$$\sum_{n=1}^{N} \frac{1}{a_n} = \sum_{n=1}^{N} \left( \frac{1}{a_n - 1} - \frac{1}{a_n + 1} \right) = \frac{1}{a_1 - 1} - \frac{1}{a_{N+1} - 1} = \frac{1}{x - 1} - \frac{1}{a_{N+1} - 1}.$$

To prove the claim, it suffices to show that the last term goes to 0 as $N \to \infty$, or equivalently, that $(\star) \lim_{n \to \infty} a_n = \infty$.

**Proof of $(\star)$:** From the given recurrence, we get $a_{n+1} - a_n = (a_n - 1)^2 \geq 0$ for all $n$. Hence the sequence $a_1, a_2, \ldots$ is non-decreasing, and we have $a_n \geq a_1 = x$ for all $n$. Using the latter inequality we get $a_{n+1} - 1 = a_n(a_n - 1) \geq x(a_n - 1) \geq x^n(a_1 - 1) = x^n(x - 1)$ for all $n$. Since $x > 1$, the right-hand side tends to infinity as $n \to \infty$, so we obtain $\lim_{n \to \infty} a_n = \infty$ as claimed.

5. Suppose that the sequence $a_1, a_2, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

**Solution.** For $k = 0, 1, 2, \ldots$ let

$$S_k = \sum_{n=2^k}^{2^{k+1}-1} a_n.$$

From the given inequality on $a_n$ we get

$$0 < S_k \leq \sum_{n=2^k}^{2^{k+1}-1} (a_{2n} + a_{2n+1}) = \sum_{m=2^{k+1}}^{2^{k+1}-1} a_m = S_{k+1}.$$

Thus, the terms $S_k$ are positive and non-decreasing. It follows that, for any integer $K \geq 1$,

$$\sum_{n=1}^{2^{K+1}-1} a_n = \sum_{k=0}^{K} S_k \geq (K + 1)S_0 = (K + 1)a_1.$$

Hence the partial sums of the series $\sum_{n=1}^{\infty} a_n$ are unbounded, so the series diverges.
6. Suppose \(a_1, a_2, a_3, \ldots\) is a sequence of positive integers such that \(a_{n+1}\) is obtained from \(a_n\) by attaching an arbitrary digit except 9 to the right of \(a_n\). (Examples of such sequences are 1, 11, 113, 11317, 113173, \ldots and 2, 20, 201, 2014, 20148, 201483, \ldots.)

Prove that any such sequence must contain infinitely many composite numbers.

**Solution.** We argue by contradiction. Suppose there is such a sequence that contains only finitely many composite numbers. Without loss of generality, we may assume that all terms in this sequence are prime numbers greater than 5. (Otherwise, remove finitely many initial terms and re-index the sequence.)

First note that attaching one of the digits 0, 2, 4, 5, 6, 8 generates a composite number, so the digits attached must be either 1, 3, or 7. (By assumption, the digit 9 is not allowed.)

Next, by the divisibility test for 3, attaching 1 or 7 increases the remainder modulo 3 by 1, while attaching 3 does not change the remainder modulo 3. Thus, after at most three attachments of 1 or 7 we obtain an integer that is divisible by 3 and hence composite. Hence, at most 3 of the attached digits can be 1 or 7, and so from some point onwards only the digit 3 can be attached.

It remains to show that in this case we are also forced to get composite numbers.

Let \(N\) be large enough such that all terms \(a_n\) with \(n \geq N\) are obtained by attaching 3’s. Then, for any \(k = 1, 2, \ldots\), we have

\[
a_{N+k} = 10^k a_N + \sum_{i=0}^{k-1} 3 \cdot 10^i = 10^k a_N + \frac{10^k - 1}{3}.
\]

We claim that for \(k = a_N - 1\), \(a_{N+k}\) is divisible by \(a_N\).

Since, by assumption, \(a_N\) is a prime > 5 (and hence coprime with 2 and 5), Fermat’s Theorem gives \(10^{a_N - 1} \equiv 1 \mod a_N\), so \(a_N\) divides \(10^{a_N - 1} - 1\), and since \(a_N > 3\), it also divides \((10^{a_N - 1} - 1)/3\). Thus, for \(k = a_N - 1\), \(a_N\) is a divisor of the right-hand of (1). Hence \(a_N\) divides the left side as well, i.e., it divides \(a_{N+a_N-1}\). Therefore \(a_{N+a_N-1}\) cannot be prime, contradicting our assumption. This completes the proof.