2016 UI UNDERGRADUATE MATH CONTEST
Solutions

1. Given positive integers n and k, let $f_k(n)$ be the number of ordered k-tuples $(a_1, a_2, \dots, a_k)$ of positive integers such that $n = a_1 \cdot a_2 \cdots a_k$. For example, $f_2(10) = 4$ since there are 4 pairs (a_1, a_2) of positive integers with product 10: $(1, 10), (2, 5), (5, 2), (10, 1)$.

   (a) Find $f_2(2016)$. (Note that $2016 = 2^5 \cdot 3^2 \cdot 7^1$.)

   (b) Find, with proof, a simple formula for $f_k(2015)$, where k is an arbitrary positive integer. (Note that $2015 = 5 \cdot 13 \cdot 31$.)

   Solution. (a) Since $2016 = 2^5 \cdot 3^2 \cdot 7$, we have $2016 = a_1 a_2$ if and only if a_1 is of the form $(*)$ $a_1 = 2^\alpha 3^\beta 7^\gamma$, where $\alpha \in \{0, 1, \dots, 5\}$, $\beta \in \{0, 1, 2\}$, and $\gamma \in \{0, 1\}$. There are 6 choices for α, 3 for β, and 2 for γ, so the total number of integers a_1 of the form $(*)$ is $6 \cdot 3 \cdot 2 = \boxed{36}$.
   (b) Since $2015 = 5 \cdot 13 \cdot 31$, we have $2015 = a_1 a_2 \cdots a_k$ if and only if each of the three prime factors $5, 13, 31$ is a prime factor of exactly one a_i, and the a_i's do not contain any prime factors other than those three. Since there are k^3 ways to distribute the three prime factors among the a_i's, we have $f_k(2015) = \boxed{k^3}$.

2. Given two positive integers n and m, call m a *descendant* of n if m can be obtained from n by replacing zero or more of its non-zero digits (in decimal representation) by 0. For example, the number 213 has 7 *non-zero* descendents: $213, 210, 203, 013 (= 13), 200, 010 (= 10), 003 (= 3)$.

   Prove that any positive integer containing exactly 2016 digits in its decimal representation and *none of whose digits is zero* has a *non-zero* descendant that is divisible by 2016.

   Solution. Let N be the given integer, and consider the integers N_k, $k = 0, 1, \dots, 2015$, obtained from N by replacing the last k digits by 0's. (In particular, $N_0 = N$ is the given integer.) Since N has 2016 digits, none of which is 0, the integers N_k are all *distinct, non-zero* descendants of N_0.
   If one of the integers $N_0, N_1, \dots, N_{2015}$ is divisible by 2016, we have obtained the desired conclusion.
   Now consider the remaining case, i.e., the case when none of the integers N_k is congruent to 0 modulo 2016. Since there are 2016 such integers and 2015 *non-zero* congruence classes modulo 2016, by the pigeonhole principle two of these integers, say N_h and N_k with $h < k$, must fall into the same congruence class modulo 2016. Then $N_h - N_k$ is divisible by 2016. But $N_h - N_k$ is the integer obtained from N by replacing the last h and the first $2016 - k$ digits by 0's (with the remaining $k - h$ digits being non-zero), and hence is a non-zero descendant of N. Thus the desired conclusion holds in this case as well.

3. Let C_1 be the unit circle $x^2 + y^2 = 1$, and let C_r denote the circle $x^2 + y^2 = r^2$, where r is a given real number with $0 < r < 1$. Two points P and Q are chosen randomly and independently on the circumference of C_1. Find, with proof, the probability that the line segment PQ intersects the circle C_r.

   Solution. Let O denote the center of the two circles, M the midpoint of the line segment PQ, and α the angle at O of the triangle POQ. Then PQ intersects the circle C_r if and only if $r \geq |OM|$. Now, $|OM| = \cos(\alpha/2)$, so the latter condition is equivalent to $r \geq \cos(\alpha/2)$, or $\alpha \geq 2\arccos(r)$.

Now the angle α represents the length of the shorter of the arcs between P and Q on the unit circle, and since Q is a random point on this circle, the angle α is uniformly distributed between 0 and π. Hence

$$P(\alpha \geq 2\arccos(r)) = \frac{1}{\pi}(\pi - 2\arccos(r)) = \frac{1}{\pi}(\pi - 2\arccos(r)) = \boxed{\frac{2}{\pi}\arcsin(r)}$$

4. Given a real number x such that $x > 1$, define a sequence $a_1, a_2, a_3, \ldots$ by $a_1 = x$, and

$$a_{n+1} = a_n^2 - a_n + 1 \quad (n = 1, 2, 3, \ldots).$$

Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and find its value, as a function of x.

Solution. We claim that the series converges, with sum $1/(x-1)$.
Writing the given recurrence as $a_{n+1} - 1 = a_n(a_n - 1)$, we get

$$\frac{1}{a_{n+1} - 1} = \frac{1}{a_n(a_n - 1)} = \frac{1}{a_n - 1} - \frac{1}{a_n}.$$

Hence, for any positive integer N we have

$$\sum_{n=1}^{N} \frac{1}{a_n} = \sum_{n=1}^{N} \left(\frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1} \right) = \frac{1}{a_1 - 1} - \frac{1}{a_{N+1} - 1} = \frac{1}{x - 1} - \frac{1}{a_{N+1} - 1}.$$

To prove the claim, it suffices to show that the last term goes to 0 as $N \to \infty$, or equivalently, that $(*)$ $\lim_{n\to\infty} a_n = \infty$.
Proof of $(*)$*:* From the given recurrence, we get $a_{n+1} - a_n = (a_n - 1)^2 \geq 0$ for all n. Hence the sequence $a_1, a_2, \ldots$ is non-decreasing, and we have $a_n \geq a_1 = x$ for all n. Using the latter inequality we get $a_{n+1} - 1 = a_n(a_n - 1) \geq x(a_n - 1) \geq \cdots \geq x^n(a_1 - 1) = x^n(x - 1)$ for all n. Since $x > 1$, the right-hand side tends to infinity as $n \to \infty$, so we obtain $\lim_{n\to\infty} a_n = \infty$ as claimed.

5. Suppose that the sequence $a_1, a_2, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Solution. For $k = 0, 1, 2, \ldots$ let

$$S_k = \sum_{n=2^k}^{2^{k+1}-1} a_n.$$

From the given inequality on a_n we get

$$0 < S_k \leq \sum_{n=2^k}^{2^{k+1}-1} (a_{2n} + a_{2n+1}) = \sum_{m=2^{k+1}}^{2^{k+1}-1} a_m = S_{k+1}.$$

Thus, the terms S_k are positive and non-decreasing. It follows that, for any integer $K \geq 1$,

$$\sum_{n=1}^{2^{K+1}-1} a_n = \sum_{k=0}^{K} S_k \geq (K+1)S_0 = (K+1)a_1.$$

Hence the partial sums of the series $\sum_{n=1}^{\infty} a_n$ are unbounded, so the series diverges.

6. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of positive integers such that a_{n+1} is obtained from a_n by attaching an arbitrary digit *except* 9 to the right of a_n. (Examples of such sequences are $1, 11, 113, 1131, 11317, 113173, \ldots$ and $2, 20, 201, 2014, 20148, 201483, \ldots$.)

   Prove that any such sequence must contain infinitely many composite numbers.

   Solution. We argue by contradiction. Suppose there is such a sequence that contains only finitely many composite numbers. Without loss of generality, we may assume that *all* terms in this sequence are prime numbers greater than 5. (Otherwise, remove finitely many initial terms and re-index the sequence.)

   First note that attaching one of the digits $0, 2, 4, 5, 6, 8$ generates a composite number, so the digits attached must be either 1, 3, or 7. (By assumption, the digit 9 is not allowed.)

   Next, by the divisibility test for 3, attaching 1 or 7 increases the remainder modulo 3 by 1, while attaching 3 does not change the remainder modulo 3. Thus, after at most three attachments of 1 or 7 we obtain an integer that is divisible by 3 and hence composite. Hence, at most 3 of the attached digits can be 1 or 7, and so from some point onwards only the digit 3 can be attached. It remains to show that in this case we are also forced to get composite numbers.

   Let N be large enough such that all terms a_n with $n \geq N$ are obtained by attaching 3's. Then, for any $k = 1, 2, \ldots$, we have

   $$a_{N+k} = 10^k a_N + \sum_{i=0}^{k-1} 3 \cdot 10^i = 10^k a_N + \frac{10^k - 1}{3}. \tag{1}$$

   We claim that for $k = a_N - 1$, a_{N+k} is divisible by a_N.

   Since, by assumption, a_N is a prime > 5 (and hence coprime with 2 and 5), Fermat's Theorem gives $10^{a_N-1} \equiv 1 \bmod a_N$, so a_N divides $10^{a_N-1} - 1$, and since $a_N > 3$, it also divides $(10^{a_N-1} - 1)/3$. Thus, for $k = a_N - 1$, a_N is a divisor of the right-hand of (1). Hence a_N divides the left side as well, i.e., it divides a_{N+a_N-1}. Therefore a_{N+a_N-1} cannot be prime, contradicting our assumption. This completes the proof.