2016 UI UNDERGRADUATE MATH CONTEST

1. Given positive integers n and k, let $f_k(n)$ be the number of ordered k-tuples $(a_1, a_2, \ldots, a_k)$ of positive integers such that $n = a_1 \cdot a_2 \cdots a_k$. For example, $f_2(10) = 4$ since there are 4 pairs (a_1, a_2) of positive integers with product 10: $(1, 10), (2, 5), (5, 2), (10, 1)$.

   (a) Find $f_2(2016)$. (Note that $2016 = 2^5 \cdot 3^2 \cdot 7^1$.)

   (b) Find, with proof, a simple formula for $f_k(2015)$, where k is an arbitrary positive integer. (Note that $2015 = 5 \cdot 13 \cdot 31$.)

2. Given two positive integers n and m, call m a *descendant* of n if m can be obtained from n by replacing zero or more of its non-zero digits (in decimal representation) by 0. For example, the number 213 has 7 *non-zero* descendents: $213, 210, 203, 013 (= 13), 200, 010 (= 10), 003 (= 3)$.

   Prove that any positive integer containing exactly 2016 digits in its decimal representation and *none of whose digits is zero* has a *non-zero* descendant that is divisible by 2016.

3. Let C_1 be the unit circle $x^2 + y^2 = 1$, and let C_r denote the circle $x^2 + y^2 = r^2$, where r is a given real number with $0 < r < 1$. Two points P and Q are chosen randomly and independently on the circumference of C_1. Find, with proof, the probability that the line segment PQ intersects the circle C_r.

4. Given a real number x such that $x > 1$, define a sequence $a_1, a_2, a_3, \ldots$ by $a_1 = x$, and

   $$a_{n+1} = a_n^2 - a_n + 1 \quad (n = 1, 2, 3, \ldots).$$

   Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and find its value, as a function of x.

5. Suppose that the sequence $a_1, a_2, \ldots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

6. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of positive integers such that a_{n+1} is obtained from a_n by attaching an arbitrary digit *except* 9 to the right of a_n. (Examples of such sequences are $1, 11, 113, 1131, 11317, 113173, \ldots$ and $2, 20, 201, 2014, 20148, 201483, \ldots$.)

   Prove that any such sequence must contain infinitely many composite numbers.

[Solutions at `http://www.math.uiuc.edu/contests.html`]