2015 UI UNDERGRADUATE MATH CONTEST
Solutions

1. Let b be a positive integer greater than 1. We say that a positive integer n is **special** in base b if it has the following two properties: (i) The base b representation of n contains each of the digits $0, 1, \ldots, b-1$ exactly once. (ii) For each $k = 1, 2, \ldots, b$, the number obtained by truncating n to its first k base b digits is divisible by k.

   For example, in base $b = 4$ the number 3210_4 is special, since (i) it contains each of the digits $0, 1, 2, 3$ exactly once, and (ii) the truncated numbers $3_4 = 3$, $32_4 = 3 \cdot 4 + 2 = 14$, $321_4 = 3 \cdot 16 + 2 \cdot 4 + 1 = 57$ and $3210_4 = 3 \cdot 64 + 2 \cdot 16 + 1 \cdot 4 + 0 = 228$ are divisible by $1, 2, 3, 4$, respectively. On the other hand, 1032_4 is not special, since $103_4 = 1 \cdot 16 + 3 = 19$ is not divisible by 3.

   (a) Show that there are **no** special numbers in base $b = 5$.

   (b) Find, with proof, **all** special numbers in base $b = 6$.

   Solution. (a) We argue by contradiction. Suppose $n = (a_4a_3a_2a_1a_0)_5 = a_4 \cdot 5^4 + \cdots + a_1 \cdot 5 + a_0$ is a special number in base 5. Then 5 must divide $a_4 \cdot 5^4 + \cdots + a_1 \cdot 5 + a_0$, so $a_0 = 0$. Moreover, 4 must divide $a_4 \cdot 5^4 + \cdots + a_1 \cdot 5$, and since $5^k \equiv 1^k = 1 \bmod 4$ for any positive integer k, it follows that 4 also divides $a_4 + \cdots + a_1$. But since the a_i's are a permutation of $0, 1, \ldots, 4$ and $a_0 = 0$, we have $a_4 + \cdots + a_1 = 1 + 2 + 3 + 4 = 10$, which is not divisible by 4. Thus we have arrived at a contradiction.

   (b) We claim that the special numbers in base 6 are the two numbers $\boxed{(543210)_6 \text{ and } (143250)_6}$.

   For the proof, suppose $n = (a_5a_4 \ldots a_0)_6 = a_5 \cdot 6^5 + a_4 \cdot 6^4 + \cdots + a_0$ is a special number in base 6. Then 6 divides n, so $a_0 = 0$. Moreover, 3 divides $(a_5a_4a_3)_6 = a_5 \cdot 6^2 + a_4 \cdot 6 + a_3$, so $a_3 = 3$. Similarly, 2 divides $(a_5a_4)_6 = a_5 \cdot 6 + a_4$, so a_2 must be even and hence either 2 or 4. Moreover, 4 divides $(a_5a_4a_3a_2)_6 = a_5 \cdot 6^3 + a_4 \cdot 6^2 + a_3 \cdot 6 + a_2$, so 4 must divide $6a_3 + a_2 = 18 + a_2$. Hence $a_2 = 2$ and $a_4 = 4$. Thus, n must be of the form $(*432*0)_6$, with the asterisks denoting the remaining two positions in the base 6 expansion of n. By property (i), these positions must be filled by the two digits 5 and 1, in some order, so n must be one of the two numbers $(543210)_6$ and $(143250)_6$. A direct check shows that both of these numbers are indeed special, so those are the only special numbers in base 6.

2. Let

   $$f(x) = \frac{x-1}{x+1},$$

   and let $f_k(x)$ be the k-th iterate of $f(x)$ defined by $f_1(x) = f(x)$ and $f_k(x) = f(f_{k-1}(x))$ for $k = 2, 3, 4, \ldots$. Find, with proof, $f_{2015}(2015)$.

   Solution. A direct calculation shows $f_2(x) = -1/x$, $f_3(x) = (1+x)/(1-x)$, and $f_4(x) = x$. Thus $f_k(x)$ is periodic in k with period 4, and therefore $f_{2015}(2015) = f_3(2015) = (1+2015)/(1-2015) = \boxed{-2016/2014}$.

3. Let $H_n = \sum_{k=1}^{n} 1/k$. Show that the infinite series

   $$\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)}$$

   converges and find its value.

   Solution. For any positive integer N we have

   (1)
   $$\begin{aligned}
   \sum_{n=1}^{N} \frac{H_{n+1}}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{H_{n+1}}{n} - \frac{H_{n+1}}{n+1} \right) \\
   &= \sum_{n=1}^{N} \left(\frac{H_n + 1/(n+1)}{n} - \frac{H_{n+1}}{n+1} \right) \\
   &= \sum_{n=1}^{N} \left(\frac{H_n + 1}{n} - \frac{H_{n+1} + 1}{n+1} \right) \\
   &= \frac{H_1 + 1}{1} - \frac{H_{N+1} + 1}{N+1} = 2 - \frac{H_{N+1} + 1}{N+1}.
   \end{aligned}$$

Now,

$$\frac{H_{N+1}}{N+1} = \frac{1}{N+1}\sum_{k=1}^{N+1}\frac{1}{k} \le \frac{1}{N+1} + \frac{1}{N+1}\int_1^{N+1}\frac{1}{x}dx = \frac{1}{N+1} + \frac{\ln(N+1)}{N+1},$$

which goes to 0 as $N \to \infty$. Hence the partial sums (1) converge as $N \to \infty$, with limit 2. Therefore the given infinite series converges and has sum 2.

4. Given any nonempty, finite set S of integers, let $f(S) = \prod_{s\in S}(s-1)$. Thus, for example, $f(\{3\}) = 2$, $f(\{3,14\}) = 2\cdot 13$, $f(\{3,14,159\}) = 2\cdot 13\cdot 158$. Find, with proof, the sum

$$\sum_{S\subseteq\{1,2,\dots,2015\}} f(S),$$

where S runs over all *nonempty* subsets of $\{1,2,\dots,2015\}$.

Solution. More generally, let

$$F(n) = \sum_{S\subseteq\{1,2,\dots,n\}} f(S).$$

We will show by induction that $(*)$ $F(n) = n! - 1$ holds for all positive integers n.

In the base case $n = 1$, the above sum reduces to $f(1) = 1 - 1 = 0$, so $F(1) = 0$, and $(*)$ holds in this case. Now suppose that $(*)$ holds for some positive integer n, and consider the nonempty subsets of $S \subseteq \{1,2,\dots,n,n+1\}$. Such a subset must be of exactly one of the forms (i) S', (ii) $S'\cup\{n+1\}$, and (iii) $\{n+1\}$, where S' is a nonempty subset of $\{1,2,\dots,n\}$. Thus,

$$\begin{aligned}
F(n+1) &= \sum_{S\subseteq\{1,2,\dots,n,n+1\}} f(S)\\
&= \sum_{S'\subseteq\{1,2,\dots,n\}} f(S') + \sum_{S'\subseteq\{1,2,\dots,n\}} f(S'\cup\{n+1\}) + f(\{n+1\})\\
&= \sum_{S'\subseteq\{1,2,\dots,n\}} f(S') + \sum_{S'\subseteq\{1,2,\dots,n\}} f(S')n + n\\
&= (n+1)\sum_{S'\subseteq\{1,2,\dots,n\}} f(S') + n\\
&= (n+1)(n!-1) + n = (n+1)! - 1,
\end{aligned}$$

which proves $(*)$ for $n+1$ and completes the induction.

5. Find, with proof, the value of the integral

$$\int_0^1 \left(\sqrt[2015]{1-x^{2014}} - \sqrt[2014]{1-x^{2015}}\right)dx.$$

Solution. We claim that the integral is equal to 0. Setting $f(x) = (1-x^{2014})^{1/2015}$ and $g(x) = (1-x^{2015})^{1/2014}$, we need to prove that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx. \qquad (1)$$

Note that $g(f(x)) = x$, so the functions f and g are inverses of each other. Also, $f(0) = 1$ and $f(1) = 0$. Substituting $x = f(y)$ and integrating by parts, we then get

$$\begin{aligned}
\int_0^1 g(x)dx &= \int_1^0 g(f(y))f'(y)dy = -\int_0^1 g(f(y))f'(y)dy\\
&= -\int_0^1 yf'(y)dy = -yf(y)\Big|_0^1 + \int_0^1 f(y)dy\\
&= \int_0^1 f(y)dy.
\end{aligned}$$

This proves (1) as desired.

6. Call a sequence of distinct points $P_1, P_2, P_3, \dots$ in the plane **well-spaced** if the distance between any two points P_i and P_j with $i \neq j$ is at least 1. Determine, with proof, the *exact* set of positive real numbers α such that, for any well-spaced sequence $\{P_n\}$ that does not contain the origin, the series

$$\sum_{n=1}^{\infty} \frac{1}{|P_n|^\alpha}$$

converges. Here $|P_n|$ denotes the distance of P_n to the origin.

Solution. We claim that the desired set is the set of real numbers α with $\alpha > 2$.

(i) We first show that the condition $\alpha > 2$ is necessary. To this end, it suffices to construct a sequence of points that are well-spaced and for which the above series diverges when $\alpha = 2$. We claim that the points of the form $P_{m,n} = (m, n)$, with $n = 1, 2, \dots$ and $m = 1, 2, \dots, n$, have this property. Since the points $P_{m,n}$ have integer coordinates and are distinct, they are well-spaced. Moreover, for fixed n, we have

$$\sum_{m=1}^{n} \frac{1}{|P_{m,n}|^2} \geq \sum_{m=1}^{n} \frac{1}{n^2} = \frac{1}{n},$$

and summing over n we get

$$\sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{|P_{m,n}|^2} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Thus the series in the problem diverges when $\alpha = 2$ (and hence also for any $\alpha < 2$). Therefore α must satisfy $\alpha > 2$.

(ii) Next we show that the condition $\alpha > 2$ is also sufficient. Let $\alpha > 2$ and a well-spaced sequence of points $\{P_i\}$ be given. Since there can be at most finitely many points with $|P_i| < 1$, we may assume without loss of generality that $|P_i| \geq 1$ for all i.

For each positive integer n, we let $f(n)$ denote the number of points P_i satisfying $n/2 \leq |P_i| < (n+1)/2$. Then

$$\sum_{i=1}^{\infty} \frac{1}{|P_i|^\alpha} \leq \sum_{n=1}^{\infty} \frac{f(n)}{(n/2)^\alpha}, = 2^\alpha \sum_{n=1}^{\infty} \frac{f(n)}{n^\alpha}.$$

Since $\alpha > 2$, to prove the convergence of the latter series, it suffices to show that

(1) $$f(n) \leq Cn \quad \text{for some constant } C \text{ and all } n = 1, 2, 3, \dots.$$

For the proof of (1), let, for each i, D_i denote the disk of radius $1/2$ centered at P_i. Since the points P_i are well-spaced, these disks are pairwise disjoint. Moreover, for each n, the disks corresponding to points satisfying $n/2 \leq |P_i| < (n+1)/2$ are contained entirely in the annulus between the circles of radius $(n-1)/2$ and $(n+2)/2$, centered at the origin. The area of this annulus is $\pi((n+2)^2 - (n-1)^2)/4 = \pi(6n+3)/4 < (3\pi)n$. Since each disk has area $\pi/4$ and the disks are nonoverlapping, it follows that there can be at most $(3\pi)n/(\pi/4) = 12n$ points P_i satisfying $n/2 \leq |P_i| < (n+1)/2$. This proves (1) with $C = 12$ and completes the proof of our claim.