2015 UI UNDERGRADUATE MATH CONTEST

1. Let b be a positive integer greater than 1. We say that a positive integer n is **special** in base b if it has the following two properties: (i) The base b representation of n contains each of the digits $0, 1, \dots, b-1$ exactly once. (ii) For each $k = 1, 2, \dots, b$, the number obtained by truncating n to its first k base b digits is divisible by k.

   For example, in base $b = 4$ the number 3210_4 is special, since (i) it contains each of the digits $0, 1, 2, 3$ exactly once, and (ii) the truncated numbers $3_4 = 3$, $32_4 = 3 \cdot 4 + 2 = 14$, $321_4 = 3 \cdot 16 + 2 \cdot 4 + 1 = 57$ and $3210_4 = 3 \cdot 64 + 2 \cdot 16 + 1 \cdot 4 + 0 = 228$ are divisible by $1, 2, 3, 4$, respectively. On the other hand, 1032_4 is not special, since $103_4 = 1 \cdot 16 + 3 = 19$ is not divisible by 3.

   (a) Show that there are **no** special numbers in base $b = 5$.

   (b) Find, with proof, **all** special numbers in base $b = 6$.

2. Let
$$f(x) = \frac{x-1}{x+1},$$
   and let $f_k(x)$ be the k-th iterate of $f(x)$ defined by $f_1(x) = f(x)$ and $f_k(x) = f(f_{k-1}(x))$ for $k = 2, 3, 4, \dots$. Find, with proof, $f_{2015}(2015)$.

3. Let $H_n = \sum_{k=1}^{n} 1/k$. Show that the infinite series
$$\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)}$$
   converges and find its value.

4. Given any nonempty, finite set S of integers, let $f(S) = \prod_{s \in S}(s-1)$. Thus, for example, $f(\{3\}) = 2$, $f(\{3, 14\}) = 2 \cdot 13$, $f(\{3, 14, 159\}) = 2 \cdot 13 \cdot 158$. Find, with proof, the sum
$$\sum_{S \subseteq \{1,2,\dots,2015\}} f(S),$$
   where S runs over all *nonempty* subsets of $\{1, 2, \dots, 2015\}$.

5. Find, with proof, the value of the integral
$$\int_0^1 \left(\sqrt[2015]{1 - x^{2014}} - \sqrt[2014]{1 - x^{2015}} \right) dx.$$

6. Call a sequence of distinct points $P_1, P_2, P_3, \dots$ in the plane **well-spaced** if the distance between any two points P_i and P_j with $i \neq j$ is at least 1. Determine, with proof, the *exact* set of positive real numbers α such that, for any well-spaced sequence $\{P_n\}$ that does not contain the origin, the series
$$\sum_{n=1}^{\infty} \frac{1}{|P_n|^{\alpha}}$$
   converges. Here $|P_n|$ denotes the distance of P_n to the origin.

[Solutions at `http://www.math.uiuc.edu/contests.html`]