2014 UI UNDERGRADUATE MATH CONTEST

Solutions

1. (a) Does there exist a *multiple of* 2013 whose decimal representation ends in the digits 2014? Explain!

   (b) Does there exist a *power of* 2 whose decimal representation ends in the digits 2014? Explain!

   Solution. The problem reduces to the question whether the congruences (1) $2013x \equiv 2014 \bmod 10000$ resp. (2) $2^x \equiv 2014 \bmod 10000$ have a positive integer solution x.

   (a) Since 2013 has no common prime factor with 10000, (1) has a solution (by a standard result in elementary number theory), so the answer to (a) is YES.

   (b) Obviously, $x = 1$ does not satisfy (2), so any solution x to (2) must be ≥ 2. Hence 2^x and 10000 must both be divisible by 4. But since $2014 = 2 \cdot 1007$ is not divisible by 4, this contradicts (2). Hence the answer to (b) is NO.

2. Let

$$S(n) = \sum_{m=1}^{n} \frac{1}{\langle \sqrt{m} \rangle},$$

   where $\langle x \rangle$ denotes the integer closest to x. (For example, $\langle \sqrt{2} \rangle = \langle 1.414\ldots \rangle = 1$ and $\langle \sqrt{3} \rangle = \langle 1.732\ldots \rangle = 2$.) Find, with proof, a general formula for $S(n^2)$.

   Solution. We will show that $(*)$ $S(n^2) = 2n - 1$.

   Given a positive integer k, we have

$$\begin{aligned} \langle \sqrt{m} \rangle = k &\Leftrightarrow k - \frac{1}{2} \leq \sqrt{m} < k + \frac{1}{2} \\ &\Leftrightarrow k^2 - k + 1/4 \leq m < k^2 + k + 1/4 \\ &\Leftrightarrow k^2 - k + 1 \leq m \leq k^2 + k \end{aligned}$$

   There are exactly $2k$ integers m satisfying the latter inequality, so the contribution of these integers to the sum over $1/\langle \sqrt{m} \rangle$ is $(2k)(1/k) = 2$.

   Letting $k = 1, 2, \ldots, n$, and summing up, we get

$$S(n^2 + n) = \sum_{m=1}^{n^2+n} \frac{1}{\langle \sqrt{m} \rangle} = \sum_{k=1}^{n} 2 = 2n.$$

   For the terms with $n^2 < m \leq n^2 + n$ we have $\langle \sqrt{m} \rangle = n$, and subtracting these terms from the above sum we get $S(n^2) = 2n - n(1/n) = 2n - 1$, proving $(*)$.

3. Let $f(n)$ denote the number of ordered pairs of positive integers $\leq n$ whose product is $\leq n^2/2$. In other words, $f(n)$ is the number of entries in the multiplication table for the first n positive integers that are $\leq n^2/2$. For example, the multiplication table for $n = 4$ is shown below, with entries that are $\leq 4^2/2 = 8$ boldfaced; there are 12 such entries, so we have $f(4) = 12$.

	1	2	3	4
1	**1**	**2**	**3**	**4**
2	**2**	**4**	**6**	**8**
3	**3**	**6**	9	12
4	**4**	**8**	12	16

Show that the limit $\lim_{n\to\infty} f(n)/n^2$ exists, and determine its value.

Solution. We will show that the limit exists and equals $(*)$ $(1/2)+(1/2)\ln 2$.

By definition, $f(n)$ is the number of ordered pairs (k,m) of positive integers satisfying

$$1 \le k, m \le n, \quad km \le n^2/2. \tag{1}$$

For each k with $1 \le k \le n$, consider the number of choices of m satisfying the conditions (1). If $k \le n/2$, there are n such choices, namely $m = 1, 2, \dots, n$; if $n/2 < k \le n$, there are $\lfloor n^2/(2k)\rfloor$ such choices, namely $m = 1, 2, \dots, \lfloor n^2/(2k)\rfloor$. Adding these counts, we get

$$\begin{aligned} f(n) &= \sum_{k=1}^{\lfloor n/2\rfloor} n + \sum_{k=\lfloor n/2\rfloor+1}^{n} \lfloor n^2/(2k)\rfloor \\ &= \lfloor n/2\rfloor n + \sum_{k=\lfloor n/2\rfloor+1}^{n} \left(\frac{n^2}{2k} + O(1)\right) \\ &= \frac{n^2}{2} + O(n) + \sum_{k=\lfloor n/2\rfloor+1}^{n} \frac{n^2}{2k} + O(n), \end{aligned} \tag{2}$$

where $O(1)$ resp. $O(n)$ denote quantities bounded in absolute value by 1 and n, respectively. Dividing by n^2 and letting $n \to \infty$ in (2), we get

$$\lim_{n\to\infty} \frac{f(n)}{n^2} = \frac{1}{2} + \frac{1}{2}\lim_{n\to\infty} \sum_{k=\lfloor n/2\rfloor+1}^{n} \frac{1}{k}. \tag{3}$$

The sum on the right of (3) can be written as

$$\sum_{k=\lfloor n/2\rfloor+1}^{n} \frac{1}{k/n}(1/n),$$

which is a Riemann sum for the integral $\int_{1/2}^{1}(1/x)dx$ with sample points $x_k = k/n$, $n/2 \le k \le n$, and interval width $x_{k+1} - x_k = 1/n$. Since the function $1/x$ is Riemann-integrable on the interval $[1/2, 1]$, these sums converge to the value of the integral, namely $\int_{1/2}^{1}(1/x)dx = \ln 2$. Thus, the limit on the right of (3) exists and is equal to $\ln 2$. This proves $(*)$.

4. Prove that, for any real numbers x and y in the interval $(0,1)$,

$$(x+y)^{x+y} \le (2x)^x (2y)^y.$$

Solution. Write $x = t(x+y)$, $y = (1-t)(x+y)$, so that $0 < t < 1$. Taking both sides of the given inequality to the power $1/(x+y)$ and dividing by $x+y$, the inequality is seen to be equivalent to

$$1 \le (2t)^t (2(1-t))^{1-t}. \tag{1}$$

Taking logarithms on both sides shows that (1) is equivalent to

$$\log(1/2) \le t\log t + (1-t)\log(1-t). \tag{2}$$

Let $f(t)$ denote the function on the right of (2). Then $f'(t) = \log t - \log(1-t)$, which is negative if $0 < t < 1/2$, equal to 0 at $t = 1/2$, and positive if $1/2 < t < 1$. Thus $f(t)$ is minimal at $t = 1/2$, and since $f(1/2) = (1/2)\log(1/2) + (1 - 1/2)\log(1 - 1/2) = \log(1/2)$, this gives (2), and hence the desired inequality.

5. Given positive integers n and m with $n \geq 2m$, let $f(n,m)$ be the number of binary sequences of length n (i.e., strings $a_1a_2 \ldots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. Find, with proof, a simple formula for $f(n,m)$.

Solution. Every sequence of the required form can be written as $B_1C_101B_2C_201\ldots01B_{m+1}C_{m+1}$, where each B_i is a block of 1's and each C_i a block of 0's, with empty blocks being allowed, and the sum of the lengths of the blocks B_i and C_i is $n-2m$. Moreover, the sequence is uniquely determined by the $(2m+2)$-tuple (1) $(b_1, c_1, b_2, c_2, \ldots, b_{m+1}, c_{m+1})$ where b_i and c_i denote the number of elements in the blocks B_i and C_i, respectively. Conversely, any tuple of the form (1) with nonnegative integers b_i and c_i satisfying $\sum_{i=1}^{m+1}(b_i+c_i) = (n-2m)$ determines a sequence of the required type. Hence the number of such sequences is equal to the number of ways one can write $2n-m$ as a sum of $(2m+2)$ nonnegative integers, with order taken into account. The latter problem is equivalent to counting the number of ways of choosing $2n-m$ donuts from $2m+2$ varieties, a well-known combinatorial problem whose answer is given by the binomial coefficient $\binom{a}{b}$ with $a = (n-2m)+(2m+2)-1 = n+1$ and $b = (2m+2)-1 = 2m+1$. Hence $f(n,m) = \binom{n+1}{2m+1}$.

6. Given a real number $x \in [0,1)$ with *binary* expansion $x = (0.b_1b_2b_3\ldots)_2$, let $f(x) = (0.b_1b_2b_3\ldots)_{10}$ be the number obtained when interpreting the binary expansion of x as a decimal expansion. For example, $f(1/2) = f((0.1000\ldots)_2) = (0.1000\ldots)_{10} = 1/10$; $f(3/8) = f(2^{-2}+2^{-3}) = f((0.0110\ldots)_2) = (0.0110\ldots)_{10} = 10^{-2}+10^{-3} = 11/1000$. Evaluate the integral

$$\int_0^1 f(x)dx.$$

Solution. Let I denote the integral to be computed. We split this integral at $x = 1/2$ to get

$$I = \int_0^1 f(x)dx = \int_0^{1/2} f(x)dx + \int_0^{1/2} f((1/2)+x)dx.$$

Now, if $0 \leq x < 1/2$ then x has the form $x = (0.0b_2b_3\ldots)_2$, and

$$\begin{aligned} f(x) &= (0.0b_2b_3\ldots)_{10} = \frac{1}{10}(0.b_2b_3\ldots)_{10} \\ &= \frac{1}{10}f(0.b_2b_3\ldots)_2 = \frac{1}{10}f(2x), \\ f((1/2)+x) &= (0.1b_2b_3\ldots)_{10} = \frac{1}{10} + \frac{1}{10}(0.b_2b_3\ldots)_{10} \\ &= \frac{1}{10} + \frac{1}{10}f(2x) \end{aligned}$$

Hence

$$\begin{aligned} I &= \frac{1}{10}\int_0^{1/2} (f(2x)+1+f(2x))dx \\ &= \frac{1}{20}\int_0^1 (1+2f(y))\,dy \\ &= \frac{1}{20} + \frac{I}{10}, \end{aligned}$$

and solving for I gives $I = 1/18$.