2014 UI UNDERGRADUATE MATH CONTEST

1. (a) Does there exist a *multiple of* 2013 whose decimal representation ends in the digits 2014? Explain!

   (b) Does there exist a *power of* 2 whose decimal representation ends in the digits 2014? Explain!

2. Let
$$S(n) = \sum_{m=1}^{n} \frac{1}{\langle \sqrt{m} \rangle},$$
where $\langle x \rangle$ denotes the integer closest to x. (For example, $\langle \sqrt{2} \rangle = \langle 1.414\ldots \rangle = 1$ and $\langle \sqrt{3} \rangle = \langle 1.732\ldots \rangle = 2$.) Find, with proof, a general formula for $S(n^2)$.

3. Let $f(n)$ denote the number of ordered pairs of positive integers $\le n$ whose product is $\le n^2/2$. In other words, $f(n)$ is the number of entries in the multiplication table for the first n positive integers that are $\le n^2/2$. For example, the multiplication table for $n = 4$ is shown below, with entries that are $\le 4^2/2 = 8$ boldfaced; there are 12 such entries, so we have $f(4) = 12$.

	1	2	3	4
1	**1**	**2**	**3**	**4**
2	**2**	**4**	**6**	**8**
3	**3**	**6**	9	12
4	**4**	**8**	12	16

   Show that the limit $\lim_{n\to\infty} f(n)/n^2$ exists, and determine its value.

4. Prove that, for any real numbers x and y in the interval $(0, 1)$,
$$(x + y)^{x+y} \le (2x)^x (2y)^y.$$

5. Given positive integers n and m with $n \ge 2m$, let $f(n, m)$ be the number of binary sequences of length n (i.e., strings $a_1a_2\ldots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. Find, with proof, a simple formula for $f(n, m)$.

6. Given a real number $x \in [0, 1)$ with *binary* expansion $x = (0.b_1b_2b_3\ldots)_2$, let $f(x) = (0.b_1b_2b_3\ldots)_{10}$ be the number obtained when interpreting the binary expansion of x as a decimal expansion. For example, $f(1/2) = f((0.1000\ldots)_2) = (0.1000\ldots)_{10} = 1/10$; $f(3/8) = f(2^{-2} + 2^{-3}) = f((0.0110\ldots)_2) = (0.0110\ldots)_{10} = 10^{-2} + 10^{-3} = 11/1000$. Evaluate the integral
$$\int_0^1 f(x)dx.$$

[Solutions at `http://www.math.uiuc.edu/contests.html`]