2013 UI UNDERGRADUATE MATH CONTEST

Solutions

1. Let $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for $n = 1, 2, \dots$.

   (i) Prove that the integers $a_1, a_2, \dots$ are pairwise coprime (i.e., do not have a common prime factor).

   (ii) Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and find its value.

Solution. (i) The first two terms of the sequence are $a_1 = 2$ and $a_2 = 2^2 - 2 + 1 = 3$, which are coprime. Thus, it remains to show that, for $n \geq 3$, a_n has no common prime factor with $a_{n-1}, a_{n-2}, \dots, a_1$. Iterating the given recurrence we get, for $n \geq 3$,

$$a_n - 1 = a_{n-1}(a_{n-1} - 1) = \cdots = a_{n-1}a_{n-2} \dots a_2(a_1 - 1) = a_{n-1}a_{n-2} \dots a_2.$$

It follows immediately that a_n cannot have any common prime factors with any of the numbers $a_2, \dots, a_{n-1}$. Moreover, since $a_{n-1}(a_{n-1} - 1)$ is always even (as a product of two consecutive integers), the relation $a_n = 1 + a_{n-1}(a_{n-1} - 1)$ implies that a_n is odd for $n \geq 2$, and therefore cannot have a common prime factor with $a_1 = 2$ either. This proves the integers $a_1, a_2, \dots$ are pairwise coprime.

(ii) First note that the formula $a_n = 1 + a_{n-1}a_{n-2} \dots a_2$ implies that the numbers a_n form an increasing sequence of positive integers and hence tend to infinity as $n \to \infty$.

Now apply the recurrence $a_{n+1} - 1 = a_n(a_n - 1)$ to get, for $n = 1, 2, \dots$,

$$\frac{1}{a_{n+1} - 1} = \frac{1}{a_n(a_n - 1)} = \frac{1}{a_n - 1} - \frac{1}{a_n},$$
$$\frac{1}{a_n} = \frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1}.$$

Summing the last identity over $n = 1, 2, \dots, N$ we get a telescoping sum on the right:

$$\sum_{n=1}^{N} \frac{1}{a_n} = \frac{1}{a_1 - 1} - \frac{1}{a_{N+1} - 1}.$$

As $N \to \infty$, the second term on the right goes to 0 (since $a_{N+1} \to \infty$). It follows that the infinite series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, with sum $1/(a_1 - 1) = 1/(2 - 1) = 1$.

2. Let

$$f(n) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and let $g(n) = (-1)^{f(n)}$. Find, with proof, $g(9999)$.

Solution. Observe that, for each k, $\lfloor n/k \rfloor$ counts the number of positive integers m for which $km \leq n$. Thus, the function $f(n)$ is equal to the number of pairs (k, m) of positive integers that satisfy $km \leq n$. Among these pairs, the number of those with $k \neq m$ is even since we can pair up (k, m) with (m, k). Hence, modulo 2, $f(n)$ is congruent to the number of remaining pairs in the above count, i.e., those of the form (k, k) with $k^2 \leq n$. There are $\lfloor \sqrt{n} \rfloor$ such k, so we have $f(n) \equiv \lfloor \sqrt{n} \rfloor$ modulo 2, and therefore $g(n) = (-1)^{f(n)} = (-1)^{\lfloor \sqrt{n} \rfloor}$. In particular, since $100^2 > 9999 > 99^2$, we have $\lfloor \sqrt{9999} \rfloor = 99$ and so $g(9999) = (-1)^{\lfloor \sqrt{9999} \rfloor} = (-1)^{99} = -1$.

3. Consider a regular n-gon in the plane with none of its sides vertical. Let $m_1, m_2, \ldots, m_n$ be the slopes of the n sides, in counterclockwise order. (The slope of a line segment from (x_1, y_1) to (x_2, y_2) is the ratio $\dfrac{y_2 - y_1}{x_2 - x_1}$.)

   Find, with proof, a simple formula for the sum

$$S_n = m_1 m_2 + m_2 m_3 + \cdots + m_{n-1} m_n + m_n m_1.$$

Solution. Note that $m_k = \tan\theta_k$, where θ_k is the angle formed by the kth side with respect to the positive x-axis. Now, for a regular n-gon, the angles of the sides increase by $2\pi/n$ at each step. Hence the angles θ_k must be of the form $\theta_k = \theta + 2\pi k/n$, for some θ.

From the trig identity $\tan(x - y) = (\tan x - \tan y)/(1 + \tan x \tan y)$ we get

$$\tan x \tan y = -1 + \frac{\tan x - \tan y}{\tan(x - y)}. \tag{1}$$

Applying (1) with $x = \theta_{k+1} = \theta + 2\pi(k + 1)/n$ and $y = \theta_k = \theta + 2\pi k/n$, we obtain

$$m_{k+1} m_k = \tan\theta_{k+1} \tan\theta_k = -1 + \frac{\tan\theta_{k+1} - \tan\theta_k}{\tan(\theta_{k+1} - \theta_k)} = -1 + \frac{\tan\theta_{k+1} - \tan\theta_k}{\tan 2\pi/n} \tag{2}$$

for each k. Summing (2) over $k = 1, 2, \ldots, n$ (with $m_{n+1} = m_1$ and $\theta_{n+1} = \theta_1$) we get

$$S_n = \sum_{k=1}^{n} m_{k+1} m_k = \sum_{k=1}^{n} (-1) + \frac{1}{\tan 2\pi/n} \sum_{k=1}^{n} (\tan\theta_{k+1} - \tan\theta_k) = -n,$$

since the second sum on the right telescopes with all terms cancelling out.

4. Given a positive integer n, let $e_i(n)$ denote its ith binary digit, counted from the right, and let $e_i(n) = 0$ if n has fewer than i digits. Define the **digital distance** of two positive integers n and m as $D(n, m) = \sum_{i=1}^{\infty} |e_i(n) - e_i(m)|$. For example, the digital distance of 6 and 13 is 3 by the following calculation:

$$6 = (110)_2, \quad (e_1(6), e_2(6), e_3(6), \ldots) = (0, 1, 1, 0, 0, \ldots)$$
$$13 = (1101)_2, \quad (e_1(13), e_2(13), e_3(13), \ldots) = (1, 0, 1, 1, 0, \ldots)$$
$$D(6, 13) = \sum_{i=1}^{\infty} |e_i(6) - e_i(13)| = |0 - 1| + |1 - 0| + |1 - 1| + |0 - 1| + |0 - 0| + \cdots = 3.$$

   Prove that any set of 171 integers in $\{1, 2, \ldots, 2013\}$ contains two elements that have digital distance at most 2.

Solution. Let $S = \{1, 2, \ldots, 2013\}$, and let $S_0 = \{0, 1, 2, \ldots, 2047\}$. Since $2047 = 2^{11} - 1$, S_0 consists of the nonnegative integers with 11 or fewer digits in their binary representation. Define two integers in S_0 to be **neighbors** if they are either equal or their binary represenions (padded on the left by 0's if necessary) differ in exactly one spot. Note that each element $s \in S_0$ has exactly 12 neighbors in S_0, the number s itself and the 11 numbers obtained by switching one of the binary digits of s.

Now, let T be a subset of S (and hence of S_0) with $|T| \geq 171$ elements. For each element $t \in T$, let S_t denote the set of its neighbors. By the above argument, each S_t has 12 elements, all contained in S_0. If the sets S_t, $t \in T$, were pairwise disjoint, then $2^{11} = |S_0| \geq \sum_{t \in T} |S_t| = 12|T| \geq 12 \cdot 171 = 2052 > 2^{11} = 2048$, which is a contraction. Hence two of these sets, say S_{t_1} and S_{t_2}, must have nonempty intersection. Thus, there exists an integer that is a neighbor to both t_1 and t_2, and hence has a binary representation that differs in at most one spot from both t_1 and t_2. It follows that the binary representations of t_1 and t_2 can differ in at most two spots. This implies that t_1 and t_2 have digital distance at most 2, as claimed.

5. Let $f(x)$ be a function on $[0,1]$ with a continuous second derivative satisfying $f(0)+f(1)=0$ and $|f''(x)| \le 1$ for all $x \in (0,1)$. Prove that

$$\left| \int_0^1 f(x)dx \right| \le \frac{1}{12}.$$

Solution. Write the integral as

$$I = \int_0^1 f(x)dx = \int_0^{1/2} (f(x) + f(1-x))dx = \int_0^{1/2} g(x)dx,$$

where $g(x) = f(x) + f(1-x)$. From the given conditions on f we get

$$\begin{aligned}
g(0) &= f(0) + f(1) = 0, \\
g'(x) &= f'(x) - f'(1-x) = -\int_x^{1-x} f''(y)dy \quad (0 \le x \le 1/2), \\
|g'(x)| &\le \int_x^{1-x} |f''(y)|dy \le (1-x) - x = 1 - 2x \quad (0 \le x \le 1/2), \\
g(x) &= g(0) + \int_0^x g'(y)dy = \int_0^x g'(y)dy \quad (0 \le x \le 1/2), \\
|g(x)| &\le \int_0^x |g'(y)|dy \le \int_0^x (1-2y)dy = x - x^2 \quad (0 \le x \le 1/2), \\
I &\le \int_0^{1/2} |g(x)|dx \le \int_0^{1/2} (x - x^2)dx = \frac{1}{8} - \frac{1}{24} = \frac{1}{12},
\end{aligned}$$

which is the desired bound.

6. Determine, with proof, the precise set of pairs (α, β) of positive real numbers for which the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(mn)^\alpha (m+n)^\beta}$$

converges.

Solution. We will show that the given series converges if and only if (α, β) satisfies

$$(*) \qquad 2\alpha + \beta > 2.$$

To do this we will simplify the given double series in several stages, *without changing its convergence properties*, until we obtain a series with known convergence properties, which turns out to converge if and only if $(*)$ holds.

Let S denote the given double series, and let S' denote the series S with the summation restricted to those pairs with $m \le n$. Clearly, $S' \le S$, and by symmetry we have $S \le 2S'$. Hence S converges if and only if S' converges.

Next, note that, under the restriction $m \le n$, we have $n^\beta \le (m+n)^\beta \le (n+n)^\beta = 2^\beta n^\beta$. Thus, each term in S' satisfies

$$\frac{1}{(nm)^\alpha (n+m)^\beta} \begin{cases} \le \dfrac{1}{(nm)^\alpha n^\beta}, \\ \ge 2^{-\beta} \dfrac{1}{(nm)^\alpha n^\beta}. \end{cases}$$

It follows that S' converges if and only if the sum

$$S'' = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{(nm)^{\alpha} n^{\beta}}$$

converges.

Next, write S'' as

$$S'' = \sum_{n=1}^{\infty} \frac{T_n}{n^{\alpha+\beta}},$$

where $T_n = \sum_{m=1}^{n} m^{-\alpha}$. To estimate this sum further, we distinguish three cases, $0 < \alpha < 1$, $\alpha = 1$, and $\alpha > 1$.

Suppose first that $0 < \alpha < 1$. In this case, comparing T_n with an integral shows that, for $n \geq 2$,

$$T_n \begin{cases} \geq \displaystyle\int_1^n x^{-\alpha} dx = \frac{n^{1-\alpha} - 1}{1-\alpha} \geq A_\alpha n^{1-\alpha}, \\ \leq 1 + \displaystyle\int_1^n x^{-\alpha} dx = 1 + \frac{n^{1-\alpha} - 1}{1-\alpha} \leq B_\alpha n^{1-\alpha}, \end{cases}$$

for appropriate *positive* constants A_α and B_α (depending on α, but independent of n). If follows that the sum S'' converges if and only if the corresponding sum with T_n replaced by $n^{1-\alpha}$, i.e., the sum

$$S''' = \sum_{n=1}^{\infty} \frac{n^{1-\alpha}}{n^{\alpha+\beta}} = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+\beta-1}},$$

converges. Now, S''' is a p-series with $p = 2\alpha + \beta - 1$, and it therefore converges if and only if $2\alpha + \beta - 1 > 1$. This condition is equivalent to $(*)$.

Next, assume $\alpha = 1$. In this case, a comparison with an integral gives $\ln n \leq T_n = \sum_{m=1}^{n} 1/m \leq \ln n + 1$, so for $n \geq 2$, T_n is within two positive constants of $\ln n$. As above, it follows that the sum S'' converges if and only if the sum with T_n replaced by $\ln n$ converges, i.e., if and only if the sum $\sum_{n=1}^{\infty} (\ln n) n^{-1-\beta}$ converges. But the latter sum converges for any positive number β. This is consistent with condition $(*)$ in the case $\alpha = 1$.

Finally, if $\alpha > 1$, then the original series is termwise majorized by the corresponding series with $\alpha = 1$, and since we always have convergence in the case $\alpha = 1$, the same is true for $\alpha > 1$. Again, this is consistent with condition $(*)$ in the case $\alpha > 1$.