2013 UI UNDERGRADUATE MATH CONTEST

1. Let $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for $n = 1, 2, \ldots$.

   (i) Prove that the integers $a_1, a_2, \ldots$ are pairwise coprime (i.e., do not have a common prime factor).

   (ii) Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and find its value.

2. Let
$$f(n) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor,$$
where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and let $g(n) = (-1)^{f(n)}$. Find, with proof, $g(9999)$.

3. Consider a regular n-gon in the plane with none of its sides vertical. Let $m_1, m_2, \ldots, m_n$ be the slopes of the n sides, in counterclockwise order. (The slope of a line segment from (x_1, y_1) to (x_2, y_2) is the ratio $\dfrac{y_2 - y_1}{x_2 - x_1}$.)
Find, with proof, a simple formula for the sum
$$S_n = m_1 m_2 + m_2 m_3 + \cdots + m_{n-1} m_n + m_n m_1.$$

4. Given a positive integer n, let $e_i(n)$ denote its ith binary digit, counted from the right, and let $e_i(n) = 0$ if n has fewer than i digits. Define the **digital distance** of two positive integers n and m as $D(n, m) = \sum_{i=1}^{\infty} |e_i(n) - e_i(m)|$. For example, the digital distance of 6 and 13 is 3 by the following calculation:
$$6 = (110)_2, \quad (e_1(6), e_2(6), e_3(6), \ldots) = (0, 1, 1, 0, 0, \ldots)$$
$$13 = (1101)_2, \quad (e_1(13), e_2(13), e_3(13), \ldots) = (1, 0, 1, 1, 0, \ldots)$$
$$D(6, 13) = \sum_{i=1}^{\infty} |e_i(6) - e_i(13)| = |0-1| + |1-0| + |1-1| + |0-1| + |0-0| + \cdots = 3.$$
Prove that any set of 171 integers in $\{1, 2, \ldots, 2013\}$ contains two elements that have digital distance at most 2.

5. Let $f(x)$ be a function on $[0, 1]$ with a continuous second derivative satisfying $f(0) + f(1) = 0$ and $|f''(x)| \leq 1$ for all $x \in (0, 1)$. Prove that
$$\left| \int_0^1 f(x)dx \right| \leq \frac{1}{12}.$$

6. Determine, with proof, the precise set of pairs (α, β) of positive real numbers for which the series
$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(mn)^{\alpha}(m+n)^{\beta}}$$
converges.

[Solutions at `http://www.math.uiuc.edu/contests.html`]