2012 U OF I UNDERGRADUATE MATH CONTEST

Solutions

1. For $i \geq 1$ and $j \geq 0$ let $R_{i,j}$ denote the number whose decimal representation consists of i 1's followed by j 0's, and let $R_i = R_{i,0}$; i.e.,

$$R_{i,j} = \underbrace{1\ldots1}_{i}\underbrace{0\ldots0}_{j}, \quad R_i = \underbrace{1\ldots1}_{i}.$$

   (a) Show that there exists a number of the form $R_{i,j}$ that is divisible by 2012.

   (b) Show that there exists a number of the form R_i that is divisible by 2011.

   Solution. (a) Note that for $i, j \geq 1$, $R_{i,j} = R_{i+j} - R_i$. Thus, to prove that $R_{i,j}$ is divisible by 2012 for some i and j it suffices to show that there exist distinct indices h and k such that $R_h \equiv R_k \bmod 2012$. But this follows from the pigeonhole principle, applied to the remainders $R_h \bmod 2012$, for $1 \leq h \leq 2013$.

   (b) The argument of the first part shows that there exist indices $j \geq 0$ and $i \geq 1$ such that 2011 divides $R_{j+i,j}$. Now $R_{j+i,j} = 10^j R_i$, and since 2011 has no common divisor with 10, it follows that 2011 divides R_i.

2. Consider a matrix consisting of infinitely many rows and finitely many columns defined as follows. The top row consists of an arbitrary given finite sequence of positive integers, not necessarily distinct. The subsequent rows are constructed as follows:

   Given a row with entries $a_1, a_2, \ldots, a_n$, the i-th entry in the following row is defined as the number of occurrences of the number a_i among the entries $a_1, a_2, \ldots, a_n$. For example, if the initial numbers are $1, 2, 1, 3, 3, 1, 4$ (i.e., three 1's, one 2, two 3's, and one 4), in the next row, each of the three 1's would be replaced by their count, namely 3, each of the two 3's would be replaced by their count, 2, and each of the single numbers 2 and 4 would be replaced by a 1. Continuing in this manner, we get the following matrix:

$$\begin{matrix} 1 & 2 & 1 & 3 & 3 & 1 & 4 \\ 3 & 1 & 3 & 2 & 2 & 3 & 1 \\ 3 & 2 & 3 & 2 & 2 & 3 & 2 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 \\ & & & \cdots & & & \end{matrix}$$

   Notice that, in this example, a stable state has been reached: From the fourth row onwards, all rows are identical, with 3 3's and 4 4's each. Your task is to prove that this is always the case:

   Prove that, for any finite set of initial numbers, from some point onwards, all rows must be identical.

   Solution. We derive the result through a sequence of observations.

   (1) *From the second row onwards, any entry k in a given row of that matrix must appear at least k times in that row.* This is because, by the construction of the matrix, k counts the number of occurrences of a given entry in the previous row, and for every occurrence of that entry in the previous row the corresponding entry in the given row is k.

   (2) *In any given column the entries in that column, from the second term onwards, form a non-decreasing sequence.* This is because, by (1), an entry k in a row other than the first row appears at least k times in the same row, so the entry in the same column in the following row must be at least equal to k.

   (3) *All entries in the matrix from the second row onwards are bounded from above by n.* This is because the entries in those rows represent counts of occurrences of numbers in the previous row, and hence are bounded by the total number of columns in the matrix, i.e., n.

   (4) *The entries in each column become constant from some point onwards.* Indeed, by (2) and (3), the entries in a given column, from the second row onwards, form a non-descreasing and bounded sequence, and therefore converge to a limit. Since the entries are all integers, this means that the terms of each column sequence must be constant from some point onwards.

(5) *The rows in the matrix become identical from some point onwards.* By (4) the entries in each column become stationary from some point onwards, and since there are finitely many columns, there is a point beyond which all column entries are stationary.

Observation (5) is what we had to show.

3. Let $a_1 = a_2 = a_3 = 1$, and for $n \geq 4$ define a_n recursively by

$$a_n = \frac{1 + a_{n-1}a_{n-2}}{a_{n-3}}.$$

Show that a_n is an integer for all n. (Hint: Consider $d_n = a_{n+1} - a_n$.)

Solution. Computing the first few terms of the sequence, we obtain $1, 1, 1, 2, 3, 7, 11, 26, 41, 97, 153, \ldots$. This does not seem to have any obvious pattern. However, forming the difference sequence $d_n = a_{n+1} - a_n$ as suggested in the hint yields $0, 0, 1, 1, 4, 4, 15, 15, 56, 56, \ldots$ for the first few terms of d_n. This suggests that d_n satisfies the recurrences (1) $d_{2n} = d_{2n-1}$ for $n \geq 1$ and (2) $d_{2n} = 4d_{2n-2} - d_{2n-4}$ for $n \geq 3$ The relations (1) and (2) can be proved by a routine (though tedious) induction argument.

Now (1) and (2) along with the initial conditions $(d_1, d_2, d_3, d_4) = (0, 0, 1, 1)$ clearly imply that the terms d_n are all integers. Since $a_n = a_1 + d_1 + \cdots + d_{n-1}$, it follows that the terms a_n are also integers as claimed.

Alternative solution: Applying the recurrence with $n + 2$ and $n + 3$ in place of n we obtain, for all $n \geq 2$,

$$\begin{aligned} 1 + a_n a_{n+1} &= a_{n+2}a_{n-1}, \\ a_n a_{n+3} &= 1 + a_{n+1}a_{n+2}. \end{aligned}$$

Adding these relations gives

$$\begin{aligned} a_n(a_{n+1} + a_{n+3}) &= a_{n+2}(a_{n-1} + a_{n+1}), \\ \frac{a_{n+1} + a_{n+3}}{a_{n+2}} &= \frac{a_{n-1} + a_{n+1}}{a_n}. \end{aligned}$$

Setting $q_n = (a_{n+1} + a_{n-1})/a_n$, it follows that $q_{n+2} = q_n$ for all $n \geq 2$. Hence $q_n = q_2 = (a_1 + a_3)/a_2 = 2$ for all even $n \geq 2$ and $q_n = q_3 = (a_2 + a_4)/a_3 = 3$ for all odd $n \geq 3$. In particular, q_n is an integer for all n. Since $a_{n+1} = q_n a_n + a_{n-1}$, it follows by induction that a_n is an integer for all n.

4. Let $f(x)$ be a polynomial and let $F(x) = \sum_{i=0}^{\infty} f^{(i)}(x)$, where $f^{(i)}$ is the i-th derivative of f (with $f^{(0)}(x) = f(x)$). (Note that since $f(x)$ is a polynomial, all derivatives $f^{(i)}(x)$ of sufficiently large order are identically zero, so the series is a finite series and hence well-defined.)

Show that if $f(x)$ is bounded from below, then so is $F(x)$, and the minimum value of F is greater than or equal to that of f; i.e., $\min_{x\in\mathbb{R}} F(x) \geq \min_{x\in\mathbb{R}} f(x)$.

Solution. If $f(x)$ is constant, then $F(x) = f(x)$, and the result holds trivially. Suppose therefore that $f(x)$ is a non-constant polynomial that is bounded from below. Then $f(x)$ must have even degree and its leading coefficient must be positive. Since $f'(x), f''(x), \ldots$ are polynomials of lower degree than f, the function $F(x) = f(x) + f'(x) + f''(x) + \cdots$ is a polynomial with the same leading term as f. Hence, it has even degree with positive leading coefficient and therefore is bounded from below and attains its minimum value at some point x_0. At x_0 we have $F'(x_0) = 0$, and since $F'(x) = f'(x) + f''(x) + \cdots = F(x) - f(x)$, it follows that $F(x_0) = f(x_0)$. Thus, $\min_{x\in\mathbb{R}} F(x) = F(x_0) = f(x_0) \geq \min_{x\in\mathbb{R}} f(x)$, as claimed.

5. Let $n \geq 2$ and let R_n denote the set of points $(x_1, \ldots, x_n)$ in n-dimensional space satisfying

$$x_i \geq 0 \quad (i = 1, 2, \ldots, n), \quad x_1 + x_2 + \cdots + x_n \leq 1.$$

Evaluate the n-dimensional integral

$$I_n = \int \cdots \int_{R_n} x_1(x_1 + x_2) \ldots (x_1 + x_2 + \cdots + x_n) dV.$$

Solution. The form of the integrand function and the region R_n suggests making the linear change of variables

$$y_k = x_1 + \cdots + x_k, \quad k = 1, 2, \ldots, n. \tag{1}$$

In matrix form, this can be written as $\mathbf{y} = A\mathbf{x}$, where $\mathbf{y}$ and $\mathbf{x}$ are column vectors with components y_i and x_i, respectively, and A is the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0\ldots \\ 1 & 1 & 0 & 0\ldots \\ 1 & 1 & 1 & 0\ldots \\ & \ldots & \ldots & \ldots \end{bmatrix}$$

Since A has determinant 1, the Jacobian determinant corresponding to the change of variables (1) is also 1.

Rewriting the given integral in terms of the variables y_i we get

$$I_n = \int \cdots \int_{S_n} y_1 y_2 \ldots y_n dV, \tag{2}$$

where S_n is the $(y_1, \ldots, y_n)$-region corresponding to R_n, namely

$$0 \le y_1 \le y_2 \le \cdots \le y_n \le 1.$$

The integral (2) can be evaluated by a symmetry argument: Note that the region S_n represents the part of the n-dimensional unit cube in which the variables $y_1, \ldots, y_n$ are in a particular order (namely increasing). Since the integrand function, $y_1 y_2 \ldots y_n$, is symmetric in the variables $y_1, \ldots, y_n$, integrating over any other particular ordering of these variables inside the unit cube produces the same value. Since there are $n!$ such orderings, the integral over any given ordering is $1/n!$ times the integral over the full n-dimensional unit cube. Therefore,

$$\begin{aligned} I_n &= \frac{1}{n!} \int_{y_n=0}^{1} \int_{y_{n-1}=0}^{1} \cdots \int_{y_1=0}^{1} y_1 \ldots y_{n-1} y_n dy_1 \ldots dy_{n-1} dy_n \\ &= \frac{1}{n!} \left(\int_0^1 y dy \right)^n = \frac{1}{n! 2^n}. \end{aligned}$$

6. Call a positive integer **defective** if its decimal representation does not contain all ten digits $0, 1, 2, \ldots, 9$. Thus, for example, the number 3141592653589 is defective (since it does not contain the digits 7 and 0), but the number 31415926535897932384626433832795028 is not defective (since it contains each of the digits $0, 1, \ldots, 9$).

   Let D be the set of all defective numbers. Determine, with proof, whether the series $\sum_{n \in D} \frac{1}{n}$ converges or diverges.

   Solution. We claim that the series converges. First note that $D = \bigcup_{i=0}^{9} D_i$, where D_i is the set of integers that do not contain the digit i. Thus, it suffices to show that, for each i, the series $\sum_{n \in D_i} 1/n$ converges.

   To prove this, we break the range of summation into finite intervals $I_k = 10^{k-1} \le n < 10^k$, $k = 1, 2, \ldots$, and let $S_{k,i} = \sum_{n \in I_k \cap D_i} 1/n$ denote the corresponding partial sum. Now, note that an integer $n \in I_k$ has at most k digits, and if n is also in the set D_i, then there are only 9 possible values for each of these digits. Hence the total number of elements in $I_k \cap D_i$ is at most 9^k. On the other hand, if $n \in I_k$ then $n \ge 10^{k-1}$. Thus,

$$S_{k,i} = \sum_{n \in I_k \cap D_i} \frac{1}{n} \le \frac{1}{10^{k-1}} \#\{n : n \in I_k \cap D_i\} \le \frac{9^k}{10^{k-1}},$$

   and therefore

$$\sum_{n \in D_i} \frac{1}{n} = \sum_{k=1}^{\infty} S_{k,i} \le 10 \sum_{k=1}^{\infty} \left(\frac{9}{10} \right)^k = \frac{9}{1 - 9/10} = 90 < \infty.$$

   Hence the series converges.