2012 U OF I UNDERGRADUATE MATH CONTEST

1. For $i \geq 1$ and $j \geq 0$ let $R_{i,j}$ denote the number whose decimal representation consists of i 1's followed by j 0's, and let $R_i = R_{i,0}$; i.e.,

$$R_{i,j} = \underbrace{1\ldots 1}_{i}\underbrace{0\ldots 0}_{j}, \quad R_i = \underbrace{1\ldots 1}_{i}.$$

   (a) Show that there exists a number of the form $R_{i,j}$ that is divisible by 2012.

   (b) Show that there exists a number of the form R_i that is divisible by 2011.

2. Consider a matrix consisting of infinitely many rows and finitely many columns defined as follows. The top row consists of an arbitrary given finite sequence of positive integers, not necessarily distinct. The subsequent rows are constructed as follows:

   Given a row with entries $a_1, a_2, \ldots, a_n$, the i-th entry in the following row is defined as the number of occurrences of the number a_i among the entries $a_1, a_2, \ldots, a_n$. For example, if the initial numbers are $1, 2, 1, 3, 3, 1, 4$ (i.e., three 1's, one 2, two 3's, and one 4), in the next row, each of the three 1's would be replaced by their count, namely 3, each of the two 3's would be replaced by their count, 2, and each of the single numbers 2 and 4 would be replaced by a 1. Continuing in this manner, we get the following matrix:

$$\begin{array}{ccccccc} 1 & 2 & 1 & 3 & 3 & 1 & 4 \\ 3 & 1 & 3 & 2 & 2 & 3 & 1 \\ 3 & 2 & 3 & 2 & 2 & 3 & 2 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 \\ & & & \cdots & & & \end{array}$$

   Notice that, in this example, a stable state has been reached: From the fourth row onwards, all rows are identical, with 3 3's and 4 4's each. Your task is to prove that this is always the case:

   Prove that, for any finite set of initial numbers, from some point onwards, all rows must be identical.

3. Let $a_1 = a_2 = a_3 = 1$, and for $n \geq 4$ define a_n recursively by

$$a_n = \frac{1 + a_{n-1}a_{n-2}}{a_{n-3}}.$$

   Show that a_n is an integer for all n. (Hint: Consider $d_n = a_{n+1} - a_n$.)

4. Let $f(x)$ be a polynomial and let $F(x) = \sum_{i=0}^{\infty} f^{(i)}(x)$, where $f^{(i)}$ is the i-th derivative of f (with $f^{(0)}(x) = f(x)$). (Note that since $f(x)$ is a polynomial, all derivatives $f^{(i)}(x)$ of sufficiently large order are identically zero, so the series is a finite series and hence well-defined.)

   Show that if $f(x)$ is bounded from below, then so is $F(x)$, and the minimum value of F is greater than or equal to that of f; i.e., $\min_{x\in\mathbb{R}} F(x) \geq \min_{x\in\mathbb{R}} f(x)$.

5. Let $n \geq 2$ and let R_n denote the set of points $(x_1, \ldots, x_n)$ in n-dimensional space satisfying

$$x_i \geq 0 \quad (i = 1, 2, \ldots, n), \quad x_1 + x_2 + \cdots + x_n \leq 1.$$

   Evaluate the n-dimensional integral

$$I_n = \int \cdots \int_{R_n} x_1(x_1 + x_2)\ldots(x_1 + x_2 + \cdots + x_n)\,dV.$$

6. Call a positive integer **defective** if its decimal representation does not contain all ten digits $0, 1, 2, \ldots, 9$. Thus, for example, the number 3141592653589 is defective (since it does not contain the digits 7 and 0), but the number 31415926535897932384626433832795028 is not defective (since it contains each of the digits $0, 1, \ldots, 9$).

   Let D be the set of all defective numbers. Determine, with proof, whether the series $\sum_{n\in D} \frac{1}{n}$ converges or diverges.