2011 U OF I UNDERGRAD MATH CONTEST

Solutions

1. Given distinct points $a_1 < a_2 < a_3 < \cdots < a_{100}$ on the real line, determine, with proof, the exact set of real numbers x for which the sum $\sum_{i=1}^{100} |x - a_i|$ takes its minimal value.

   Solution. Let $S(x) = \sum_{i=1}^{100} |x - a_i|$ the sum we seek to minimize. By the triangle inequality, we have, for any $i \in \{1, 2, \ldots, 100\}$,

   $$|a_i - a_{101-i}| = |(a_i - x) + (x - a_{101-i}| \le |a_i - x| + |x - a_{101-i}|. \tag{1}$$

   Summing over $i = 1, \ldots, 50$ we get

   $$\sum_{i=1}^{50} |a_i - a_{101-i}| \le S(x). \tag{2}$$

   The real numbers x that minimize $S(x)$ are exactly those x for which equality holds in (2). Since equality holds in (1) if and only if $a_i - x$ and $x - a_{101-i}$ have the same sign, i.e., if and only if x lies between a_i and a_{101-i}, equality holds in (2) if and only if x lies between a_i and a_{101-i} for each $i = 1, 2, \ldots, 50$, i.e., if and only if $a_{50} \le x \le a_{51}$. Hence the real numbers that minimize $S(x)$ are exactly those in the interval $a_{50} \le x \le a_{51}$.

2. Consider a game played on a finite sequence of positive integers in which two types of moves, A and B, are allowed: A move of type A ("Add") replaces two adjacent integers in the sequence by their sum; for example, $(\ldots, 20, 11, \ldots) \xrightarrow{A} (\ldots, 31, \ldots)$. A move of type B ("Break up") replaces a multi-digit integer in the sequence by the sequence of its *nonzero* decimal digits; for example, $(\ldots, 2011, \ldots) \xrightarrow{B} (\ldots, 2, 1, 1, \ldots)$

   The moves may be combined in any manner. For example, given the sequence $(3, 14, 159, 26)$, a possible sequence of moves is the following:

   $$(3, 14, 159, 26) \xrightarrow{A} (17, 159, 26) \xrightarrow{B} (17, 1, 5, 9, 26) \xrightarrow{A} (18, 5, 9, 26) \xrightarrow{A} (18, 5, 35) \xrightarrow{A} (18, 40) \xrightarrow{B} (18, 4) \xrightarrow{B} (1, 8, 4) \xrightarrow{A} (9, 4) \xrightarrow{A} (13) \xrightarrow{B} (1, 3) \xrightarrow{A} (4)$$

   Once the sequence is reduced to a single one-digit number, any further moves will leave it unchanged, the game terminates, and we call the final number obtained the *terminal number* of the game.

   (a) Prove that, for any finite initial sequence of positive integers, the game always terminates, regardless of the particular sequence of moves performed.

   (b) Suppose this game is played on the sequence $(1, 2, 3, 4, \ldots, 2011)$. What is the terminal number?

   Solution. (a) Given a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ of positive integers, let $s(\mathbf{a})$ denote the sum of its terms, and let $n(\mathbf{a})$ denote the number of its terms. We claim that the following properties hold:

   (i) A move of type A leaves $s(\mathbf{a})$ unchanged, but decreases $n(\mathbf{a})$.

   (ii) A move of type B decreases $s(\mathbf{a})$.

   Property (i) follows immediately from the definition of a type A move. To prove property (ii), it suffices to show that the sum of digits of an integer with 2 or more digits is strictly less than the integer itself; to see this, note that if $n = \sum_{i=0}^{r} a_i 10^i$ with $a_i \in \{0, 1, \ldots, 9\}$ and $a_r \in \{1, 2, \ldots, 9\}$, then $n \ge \sum_{i=0}^{r} a_i$, with strict inequality unless $n = a_0$, i.e., unless n is a single digit integer.

   Since each B move decreases $s(\mathbf{a})$ by at least 1, while an A move leaves $s(\mathbf{a})$ unchanged, there can be at most finitely many B moves. On the other hand, since an A move decreases the number of terms of the sequence, there can be at most finitely many *consecutive* A moves. Therefore the total number of moves must be finite.

   (b) The key observation here is that the remainder modulo 9 of the sum of all terms of the sequence is an invariant under both moves. This is obvious in the case of an A move since such a move leaves the sum unchanged. In the case of a B move this follows from the divisibility test for 9, according to which the sum of decimal digits of a positive integer is congruent to the integer itself.

Since the terminal number is a single digit positive integer, it is uniquely determined by the remainder modulo 9 of the sum of all terms of the sequence. For the given sequence $(1, 2, \ldots, 2011)$, we have

$$\sum_{i=1}^{2011} i = \frac{2011 \cdot 2012}{2} = 2011 \cdot 1006 \equiv 4 \cdot 7 = 28 \equiv 1 \bmod 9,$$

so the terminal number for this sequence is 1.

3. Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and for $n \geq 4$ define a_n to be the last digit of the sum of the preceding **three** terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) 124734419447 .... Determine, with proof, whether or not the string 1001 occurs in this sequence. (Hint: Do **not** attempt this by brute force!)

   Solution. First note that the sequence can be continued backwards in a unique manner by setting $a_{n-1} = a_{n+2} - a_{n+1} - a_n \mod 10$. Doing so, one finds that the first four terms prior to the given terms are 1, 0, 0, and 1. Thus, the string 1001 occurs in the extended sequence. To show that it also occurs in the original sequence (i.e., to the right of 1247...), note that the sequence is uniquely determined, backwards and forwards, by any three consecutive digits in the sequence. Since there are finitely many possibilities for such triples of consecutive digits, one such triple must occur again in the sequence, and the sequence is therefore is periodic (in both directions). In particular, any string that occurs somewhere in the extended sequence, occurs infinitely often and arbitrarily far out along the given sequence. Hence 1001 does occur infinitely often in this sequence. (While this term occurs immediately to the left of the given initial string 1247..., its first occurrence to the right is at the 120-th term. This would be hard to discover by a hand calculation!)

4. Let $P(x)$ be a polynomial of degree 10 satisfying $P(0) = 1, P(1) = 2^1, P(2) = 2^2, \ldots, P(10) = 2^{10}$. Determine, with proof, the value $P(11)$.

   Solution. The answer is $2^{11} - 1 = 2047$. This follows from the following more general result: If $P_n(x)$ is a polynomial satisfying $P_n(k) = 2^k$ for $k = 0, 1, \ldots, n$, then $P_n(n+1) = 2^{n+1} - 1$.

   To prove this, set $Q_n(x) = P_{n+1}(x+1) - P_{n+1}(x)$. Note that Q_n is a polynomial of degree n, with $Q_n(k) = P_{n+1}(k+1) - P_{n+1}(k) = 2^{k+1} - 2^k = 2^k = P_n(k)$ for $k = 0, 1, \ldots, n$. Since Q_n and P_n are polynomials of degree n that agree on $n+1$ pairwise distinct points, they must be identical. Thus we have $P_n(x) = Q_n(x) = P_{n+1}(x+1) - P_{n+1}(x)$ for all x. Setting $p_n = P_n(n+1)$, we then obtain the recurrence formula $p_n = p_{n+1} - 2^{n+1}$, which implies $p_{n+1} = p_n + 2^{n+1} = p_{n-1} + 2^{n+1} + 2^n = \cdots = p_0 + 2^{n+1} + 2^n + \ldots + 2^1 = p_0 + 2^{n+2} - 2$. Since $P_0(x)$ is the constant polynmial 1, we have $p_0 = 1$, giving $p_n = p_{n+1} - 2^{n+1} = 2^{n+2} - 2^{n+1} - 1 = 2^{n+1} - 1$. Thus, $P_n(n+1) = 2^{n+1} - 1$, as claimed.

5. Let f be a function from the positive integers into the positive integers and satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n. Find $f(101)$.

   Solution. We will show that, for any integers $k \geq 0$ and $0 \leq m < 3^k$,

   $$(*) \qquad f(3^k + m) = 2 \cdot 3^k + m \quad (k = 0, 1, 2, \ldots, 0 \leq m < 3^k).$$

   Since $101 = 3^4 + 20$, we obtain from $(*)$ $f(101) = f(3^4 + 20) = 2 \cdot 3^4 + 19 = \boxed{182}$.

   Proof of $(*)$**:** We first observe that the first condition on f implies $f(n+1) \geq f(n)+1$, $f(n+2) \geq f(n+1)+1 \geq f(n)+2$, ..., (1) $f(n+m) \geq f(n) + m$, any positive integers n and m.

   Next, let $a = f(1)$. If $a > 3$, then the second condition implies $f(a) = 3$ which contradicts (1) with $n = a - 1$ and $m = 1$. If $a = 1$, then we get $3 = f(f(1)) = f(a) = f(1) = a$ which is a contradiction. Finally, if $a = 3$, then we have $3 = f(f(1)) = f(3)$, whereas (1) implies $f(3) \geq f(1) + 2 = a + 2 = 5$, so we get again a contradiction. Thus, we necessarily have (2) $f(1) = 2$, which in turn implies (3) $f(2) = f(f(1)) = 3$.

   We now use induction to show that, for any nonnegative integer k, (4) $f(2 \cdot 3^k) = 3^{k+1}$ and (5) $f(3^k) = 2 \cdot 3^k$.

   For $k = 0$, (4) and (5) reduce to (2) and (3). Assume now that (4) and (5) both hold for some nonnegative integer k. Then $f(3^{k+1}) = f(f(2 \cdot 3^k)) = 3 \cdot 2 \cdot 3^k = 2 \cdot 3^{k+1}$ and $f(2 \cdot 3^{k+1}) = f(f(3^{k+1})) = 3 \cdot 3^{k+1} = 3^{k+2}$, which proves these formulas for $k + 1$ and thus completes the induction.

   From (1), (4) and (5), we see that the values $f(3^k + m)$, $m = 0, 1, \ldots 3^k - 1$, must form an increasing sequence of 3^k distinct integers, all contained in the interval $[2 \cdot 3^k, 3 \cdot 3^k - 1]$. Since there are exactly 3^k integers in that interval, these values must fill the entire interval, i.e., we have $f(3^k + m) = 3^k + m$ for $0 \leq m < 3^k$. This proves $(*)$

6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and let $A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Prove that if the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, then so does the series $\sum_{n=1}^{\infty} \frac{1}{A_n}$.

Solution. For any nonnegative integer k, set

$$S_k = \sum_{2^k \le i < 2^{k+1}} a_i, \quad T_k = \sum_{2^k \le i < 2^{k+1}} \frac{1}{a_i}.$$

By Cauchy-Schwarz,

$$2^{2k} = \left(\sum_{2^k \le i < 2^{k+1}} \sqrt{a_i} \cdot \frac{1}{\sqrt{a_i}} \right)^2 \le S_k T_k,$$

so

$$\frac{2^{2k}}{S_k} \le T_k.$$

On the other hand, for $2^{k+1} \le n < 2^{k+2}$ we have

$$\frac{1}{A_n} = \frac{n}{a_1 + a_2 + \cdots + a_n} \le \frac{n}{S_k} \le \frac{2^{k+2}}{S_k}$$

and hence

$$\sum_{2^{k+1} \le n < 2^{k+2}} \frac{1}{A_n} \le 2^{k+1} \frac{2^{k+2}}{S_k} = \frac{2^{2k+3}}{S_k} \le 2^3 T_k = 8 \sum_{2^k \le i < 2^{k+1}} \frac{1}{a_i}.$$

Summing over all integers $k \ge 0$ gives

$$\sum_{2 \le n < \infty} \frac{1}{A_n} \le 8 \sum_{1 \le i < \infty} \frac{1}{a_i}.$$

Thus, if $\sum_{i=1}^{\infty} 1/a_i$ converges, then so does the series $\sum_{n=1}^{\infty} 1/A_n$.