2011 U OF I UNDERGRAD MATH CONTEST

1. Given distinct points $a_1 < a_2 < a_3 < \cdots < a_{100}$ on the real line, determine, with proof, the exact set of real numbers x for which the sum $\sum_{i=1}^{100} |x - a_i|$ takes its minimal value.

2. Consider a game played on a finite sequence of positive integers in which two types of moves, A and B, are allowed: A move of type A ("Add") replaces two adjacent integers in the sequence by their sum; for example, $(\dots, 20, 11, \dots) \xrightarrow{\text{A}} (\dots, 31, \dots)$. A move of type B ("Break up") replaces a multi-digit integer in the sequence by the sequence of its *nonzero* decimal digits; for example, $(\dots, 2011, \dots) \xrightarrow{\text{B}} (\dots, 2, 1, 1, \dots)$

   The moves may be combined in any manner. For example, given the sequence $(3, 14, 159, 26)$, a possible sequence of moves is the following:

   $$(3,14,159,26) \xrightarrow{\text{A}} (17,159,26) \xrightarrow{\text{B}} (17,1,5,9,26) \xrightarrow{\text{A}} (18,5,9,26) \xrightarrow{\text{A}} (18,5,35) \xrightarrow{\text{A}} (18,40) \xrightarrow{\text{B}} (18,4) \xrightarrow{\text{B}} (1,8,4) \xrightarrow{\text{A}} (9,4) \xrightarrow{\text{A}} (13) \xrightarrow{\text{B}} (1,3) \xrightarrow{\text{A}} (4)$$

   Once the sequence is reduced to a single one-digit number, any further moves will leave it unchanged, the game terminates, and we call the final number obtained the *terminal number* of the game.

   (a) Prove that, for any finite initial sequence of positive integers, the game always terminates, regardless of the particular sequence of moves performed.

   (b) Suppose this game is played on the sequence $(1, 2, 3, 4, \dots, 2011)$. What is the terminal number?

3. Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and for $n \geq 4$ define a_n to be the last digit of the sum of the preceding **three** terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) 124734419447.... Determine, with proof, whether or not the string 1001 occurs in this sequence. (Hint: Do **not** attempt this by brute force!)

4. Let $P(x)$ be a polynomial of degree 10 satisfying $P(0) = 1, P(1) = 2^1, P(2) = 2^2, \dots, P(10) = 2^{10}$. Determine, with proof, the value $P(11)$.

5. Let f be a function from the positive integers into the positive integers and satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n. Find $f(101)$.

6. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, and let $A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Prove that if the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, then so does the series $\sum_{n=1}^{\infty} \frac{1}{A_n}$.

[Solutions at `http://www.math.uiuc.edu/contests.html`]