U OF I UNDERGRADUATE MATH CONTEST

MARCH 6, 2010
Solutions

1. Let $a_1, a_2, a_3, \dots$ be an infinite sequence of positive integers, and let a new sequence $q_1, q_2, q_3, \dots$ be defined by $q_1 = a_1$, $q_2 = a_2 q_1 + 1$, and $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 3$. Prove that no two consecutive q_n's are even.

   Solution. We argue by contradiction. Suppose there exist pairs of consecutive q_n's that are both even. Among these let (q_i, q_{i+1}) be the pair with smallest index i. First note that, if q_1 is even, then $q_2 = a_2 q_1 + 1$ is odd. Thus q_1 and q_2 cannot both be even, so the minimal index i such that q_i and q_{i+1} are both even must be at least 2. By the given recurrence we have $q_{i-1} = q_{i+1} - a_{i+1} q_i$, so q_{i-1} is the difference of two even numbers and therefore must itself be even. Hence (q_{i-1}, q_i) is a pair of consecutive even terms among the q_n's, contradicting the minimality of i. Thus there do not exist consecutive even members of the sequence.

2. A function $f(n)$ is defined for all positive integers n as follows: First add the digits of n (in decimal notation) to get a number n_1, say; then add the digits of n_1 to get n_2; continue this process until a single digit number is obtained; that last number (between 1 and 9) is called $f(n)$. Thus, for example, $f(989) = 8$, since $9+8+9 = 26$, $2+6 = 8$. Prove that, for all positive integers n, $f(1234567 \cdot n) = f(n)$.

   Solution. We use congruences modulo 9. By an extension of the test for divisibility by 9, any positive integer is congruent modulo 9 to the sum of its decimal digits. (Proof: If $n = a_0 + a_1 10 + a_2 10^2 + \cdots + a_k 10^k$, with $a_i \in \{0, 1, \dots, 9\}$, then modulo 9, $n \equiv a_0 + a_1 1^1 + a_2 1^2 + \cdots + a_k 1^k = a_0 + a_1 + \cdots + a_k$.) Since $f(n)$ is obtained by an iteration of the "sum of digits" function, it follows that $f(n)$ satisfies the congruence $f(n) \equiv n \bmod 9$. Moreover, since $f(n)$ is in the set $\{1, 2, \dots, 9\}$, $f(n)$ is uniquely defined by its congruence modulo 9. Thus, to prove the claim, it suffices to show that, for all positive integers n,

   $$1234567 \cdot n \equiv n \bmod 9.$$

   But the latter follows from the fact that the number 1234567 is congruent to $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 \equiv 1$ modulo 9.

3. Let α be the real number whose decimal representation is of the form $\alpha = 0.A_1 A_2 A_3 A_4 \dots$, where A_n denotes the block consisting of the last 3 digits of 2^n (padded with 0's at the beginning if necessary—for example, $A_1 = 002$, $A_2 = 004$, and $A_{10} = 024$). Determine, with proof, whether α is rational.

   Solution. The number α is rational. To prove this, we will show that the sequence of 3-digit blocks A_n, and hence the decimal expansion of α, is ultimately periodic.

   First, note that there are at most 10^3 possible choices for A_n, so by the pigeonhole principle there exist indices $n > m$ such that $A_n = A_m$. We will show that by induction that $(*)$ $A_{n+i} = A_{m+i}$ holds for all $i \geq 0$, and hence that the sequence A_i is ultimately periodic with period $p = n - m$.

   For $i = 0$ $(*)$ holds by our choice of n and m. For the proof of the induction step, observe that A_n satisfies $A_n \equiv 2^n \bmod 10^3$, and in fact is the unique integer among $\{0, 1, \dots, 999\}$ satisfying this congruence. Assume now that $(*)$ holds for some $i \geq 0$. Then

   $$A_{n+i+1} \equiv 2^{n+i+1} \equiv 2 \cdot 2^{n+i} \equiv 2A_{n+i} = 2A_{m+i} \equiv 2 \cdot 2^{m+i} = 2^{m+i+1} \equiv A_{m+i+1} \bmod 10^3.$$

Since the congruence modulo 10^3 defines A_n uniquely, this gives $(*)$ for $n = i + 1$, and completes the induction.

4. Let n be a positive integer, and let S be a set of integers in $[0, 2^n)$ such that the binary representations of any two of these integers differ in at least 3 positions. For example, if $n = 4$, then 4 and 9, but not 4 and 8 can both be in the set, since the binary representations of 4 and 9, 0100 and 1001, differ in 3 positions, but not those of 4 and 8. Show that S can contain no more than $2^n/(n + 1)$ integers.

   Solution. Each element in S can be represented in binary as a string of n 0's and 1's. Define a "neighbor" of such a string s as any string that differs from s in at most one posititition. (For example, 1100 is a neighbor of 1000, but 0100 is not.) Obviously, each element $s \in S$ has exactly $n+1$ neighbors (including s itself). Moreover, no n-digit string can be a neighbor to two distinct elements s_1, s_2 of S for otherwise s_1 and s_2 could only differ in at most two positions, contrary to the given hypothesis. Thus, the number of binary strings of length n that are neighbors of some element of S is at least $|S|(n+1)$, where $|S|$ is the number of elements of S. On the other hand, this number is at most equal to the total number of binary strings of length n, namely 2^n. Hence $|S| \leq 2^n/(n + 1)$.

5. Determine, with proof, whether the series $\sum_{n=1}^{\infty} \sin(\pi(\sqrt{n^2 + 1}))$ converges.

   Solution. We will exploit the fact that $\sqrt{n^2 + 1}$ is close to $\sqrt{n^2} = n$. To make this precise, define ϵ_n by $\sqrt{n^2 + 1} = n + \epsilon_n$. Then

$$\epsilon_n = \sqrt{n^2 + 1} - \sqrt{n^2} = \frac{(n^2 + 1) - n^2}{\sqrt{n^2 + 1} + \sqrt{n^2}} = \frac{1}{\sqrt{n^2 + 1} + \sqrt{n^2}}$$

   In particular, this shows that the numbers ϵ_n satisfy $0 < \epsilon_n < 1/2$ for all n, are monotonically decreasing, and tend to 0 as $n \to \infty$.

   The n-th term in the given series can then be written as

$$\sin(\pi(\sqrt{n^2 + 1}) = \sin(\pi n + \pi\epsilon_n) = (-1)^n \sin(\pi\epsilon_n).$$

   Since $\sin x$ is monotonically increasing in the interval $0 < x < \pi/2$, the properties of ϵ_n established above imply that the sequence of numbers $\sin(\pi\epsilon_n)$ is monotonically decreasing, with limit 0. Thus, the given series is an alternating series whose terms are monotonically decreasing to 0, and hence converges by the alternating series test.

6. Given a polynomial $P(x)$, a finite sequence of distinct numbers $a_1, \ldots, a_k$ is said to be a *cycle of length k for P* if $P(a_i) = a_{i+1}$ for $1 \leq i \leq k - 1$ and $P(a_k) = a_1$. If all a_i are integers, the cycle is called an *integer cycle.*

   Determine, with proof, all polynomials with integer coefficients that have an integer cycle of length ≥ 2. (Hint: Consider separately the case when the cycle length is ≥ 3 and the case when the cycle length is 2.)

   Solution. **Cycles of length ≥ 3:** We first show that there exist no polynomials with integer coefficients that have an integer cycle of length ≥ 3. We argue by contradiction. Suppose that $P(x)$ is a polynomial with integer coefficients and $a_1, a_2, \ldots, a_k$ is an integer cycle of length $k \geq 3$ for this polynomial.

   We use the fact that, for any distinct integers a and b, $(*)$ $(a - b)|P(a) - P(b)$, which can be seen, for example, by applying the factorization $a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1})$ to each term in the difference $P(a) - P(b)$. Applying $(*)$ with the pairs $(a, b) = (a_1, a_2), (a_2, a_3), \ldots, (a_k, a_1)$, we obtain

$$a_2 - a_1 \mid P(a_2) - P(a_1) = a_3 - a_2 \mid P(a_3) - P(a_2) \ldots$$
$$\cdots \mid P(a_k - P(a_{k-1}) = a_1 - a_k \mid P(a_1) - P(a_k) = a_2 - a_1.$$

Since the first and last terms in this chain of divisor relations are equal, we must have

$$(a_2 - a_1) = \pm(a_3 - a_2) = \cdots = \pm(a_k - a_{k-1}) = \pm(a_1 - a_k) = \pm(a_2 - a_1).$$

Moreover, if for some i with $2 \leq i \leq k-1$ we have $a_i - a_{i-1} = -(a_{i+1} - a_i)$, then $a_{i-1} = a_{i+1}$, contradicting the requirement that the a_i be distinct. Thus, we have $a_i - a_{i-1} = a_{i+1} - a_i$ for $2 \leq i \leq k-1$. This in turn implies $a_k - a_1 = (k-1)(a_2 - a_1)$, and since $k \geq 3$, this contradicts the last of the relations above, $(a_1 - a_k) = \pm(a_2 - a_1)$. Thus cycles of length ≥ 3 do not exist.

Cycles of length 2: The above argument breaks down when $k = 2$, and the conclusion in fact does not hold. For example, the polynomial $P(x) = -x$ has cycles $(1, -1)$, $(2, -2)$, $(3, -3)$, etc. The following argument determines all polynomials with integer coefficients and integer cycles of length 2: Suppose $P(x)$ is such a polynomial with cycle (a_1, a_2). Let $Q(x) = P(x) + x - a_1 - a_2$. Then Q is a polynomial with integer coefficients satisfying $Q(a_1) = P(a_1) + a_1 - a_1 - a_2 = P(a_1) - a_2 = 0$ and $Q(a_2) = P(a_2) + a_2 - a_1 - a_2 = P(a_2) - a_1 = 0$, so Q has zeros at a_1 and a_2. Hence $Q(x) = (x - a_1)(x - a_2)R(x)$, where $R(x)$ is a polynomial with integer coefficients. It follows that $P(x)$ must be of the form

$$(*) \qquad P(x) = (x - a_1)(x - a_2)R(x) - x + a_1 + a_2.$$

Conversely, any polynomial $P(x)$ of the above form satisfies $P(a_1) = a_2$ and $P(a_2) = a_1$ and thus has cycle (a_1, a_2). Therefore, the polyomials with integer coefficients that have an integer cycle of length 2 are exactly those of the form $(*)$, where a_1 and a_2 are distinct integers and $R(x)$ is a polynomial with integer coefficients.