U OF I UNDERGRADUATE MATH CONTEST

MARCH 6, 2010

1. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let a new sequence $q_1, q_2, q_3, \ldots$ be defined by $q_1 = a_1$, $q_2 = a_2 q_1 + 1$, and $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 3$. Prove that no two consecutive q_n's are even.

2. A function $f(n)$ is defined for all positive integers n as follows: First add the digits of n (in decimal notation) to get a number n_1, say; then add the digits of n_1 to get n_2; continue this process until a single digit number is obtained; that last number (between 1 and 9) is called $f(n)$. Thus, for example, $f(989) = 8$, since $9 + 8 + 9 = 26$, $2 + 6 = 8$. Prove that, for all positive integers n, $f(1234567 \cdot n) = f(n)$.

3. Let α be the real number whose decimal representation is of the form $\alpha = 0.A_1A_2A_3A_4\ldots$, where A_n denotes the block consisting of the last 3 digits of 2^n (padded with 0's at the beginning if necessary—for example, $A_1 = 002$, $A_2 = 004$, and $A_{10} = 024$). Determine, with proof, whether α is rational.

4. Let n be a positive integer, and let S be a set of integers in $[0, 2^n)$ such that the binary representations of any two of these integers differ in at least 3 positions. For example, if $n = 4$, then 4 and 9, but not 4 and 8 can both be in the set, since the binary representations of 4 and 9, 0100 and 1001, differ in 3 positions, but not those of 4 and 8. Show that S can contain no more than $2^n/(n + 1)$ integers.

5. Determine, with proof, whether the series $\sum_{n=1}^{\infty} \sin(\pi(\sqrt{n^2 + 1}))$ converges.

6. Given a polynomial $P(x)$, a finite sequence of distinct numbers $a_1, \ldots, a_k$ is said to be a *cycle of length k for P* if $P(a_i) = a_{i+1}$ for $1 \leq i \leq k - 1$ and $P(a_k) = a_1$. If all a_i are integers, the cycle is called an *integer cycle*.

   Determine, with proof, all polynomials with integer coefficients that have an integer cycle of length ≥ 2. (Hint: Consider separately the case when the cycle length is ≥ 3 and the case when the cycle length is 2.)

[Solutions at `http://www.math.uiuc.edu/contests.html`]