U OF I UNDERGRADUATE MATH CONTEST
MARCH 7, 2009
Solutions

1. Given a nonnegative integer n, let $\overleftarrow{n}$ denote the integer obtained by reversing the digits of n in the standard decimal representation; for example, $\overleftarrow{935} = 539$. Let $f(n) = n + \overleftarrow{n}$, $g(n) = n - \overleftarrow{n}$, and $h(n) = f(g(n))$. For example, if $n = 935$, then $g(n) = 935 - 539 = 396$, and $h(n) = f(396) = 396 + 693 = 1089$; if $n = 701$, then $g(n) = 701 - 107 = 594$, $h(n) = 594 + 495 = 1089$.

   Prove that these results are no accident by showing that $h(n) = 1089$ for all three digit integers n whose first digit exceeds the last digit by at least 2.

   Solution. Let n denote an integer of the given form, i.e., $n = a_2a_1a_0$ with $a_2 \geq a_0+2$. Then

$$\begin{aligned}
n &= 100a_2 + 10a_1 + a_0,\\
\overleftarrow{n} &= 100a_0 + 10a_1 + a_2,\\
g(n) &= n - \overleftarrow{n}\\
&= 100 \cdot (a_2 - a_0) - (a_2 - a_0)\\
&= 100 \cdot (a_2 - a_0 - 1) + 10 \cdot 9 + 1 \cdot (10 - a_2 + a_0),\\
\overleftarrow{g(n)} &= 100 \cdot (10 - a_2 + a_0) + 10 \cdot 9 + 1 \cdot (a_2 - a_0 - 1),\\
h(n) &= g(n) + \overleftarrow{g(n)}\\
&= 100 \cdot (10 - 1) + 10 \cdot (9 + 9) + (10 - 1) = 1089,
\end{aligned}$$

   as claimed. Note that the given condition on the first and last digits of n, namely $a_2 - a_0 \geq 2$, ensures that the cofficients $a_2 - a_0 - 1$ and $10 - a_2 + a_0$ in the above expressions are integers in the interval $[1, 9]$, so these expressions indeed represent proper decimal expansions.

2. Let S be a set of 16 distinct positive integers, all less than 60. Show that there exist four pairwise distinct elements $a, b, c, d \in S$ such that $a + b = c + d$.

   Solution. There are $\binom{16}{2} = \frac{16 \cdot 15}{2} = 120$ unordered pairs $\{a, b\}$ of distinct elements in S. Since the elements of S are integers between 1 and 60, the sums $a + b$ of these pairs are integers between 2 and 120. Since there are 119 integers in this interval and we have 120 pairs $\{a, b\}$, two (distinct) such pairs, say $\{a, b\}$ and $\{a', b'\}$, must have the same sum, i.e., $a + b = a' + b'$. To complete the proof, it remains to show that the integers a, b, a', b' involved are pairwise distinct. By construction, we have $a \neq b$ and $a' \neq b'$. If $a' = a$, then the identity $a + b = a' + b'$ implies $b' = b$, so $\{a', b'\} = \{a, b\}$, contrary to our assumption that $\{a, b\}$ and $\{a', b'\}$ represent distinct pairs. Thus, $a' \neq a$, and the same argument shows that $a' \neq b$, $b' \neq a$, and $b' \neq b$. Hence the four elements a, b, a', b' are pairwise distinct.

3. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and B the area of the region lying to the right of the y-axis and to the left of s. Prove that the sum of these areas, $A + B$, depends only on the arc length, and not on the position, of s.

Solution. [This appeared as Problem A2 in the 1998 Putnam Exam] The problem can be approached geometrically, or analytically. Here is an analytic solution:

Let θ_1 and θ_2 denote the angles bounding the given arc, so that $0 \le \theta_1 \le \theta_2 \le \pi/2$ by the assumption that the arc lies in the first quadrant. Then

$$\begin{aligned} A &= \int_{\cos\theta_2}^{\cos\theta_1} \sqrt{1-x^2}\,dx \\ &= \int_{\theta_1}^{\theta_2} \sin^2\theta\,d\theta \\ &= \int_{\theta_1}^{\theta_2} \frac{1}{2}(1-\cos(2\theta))\,d\theta \\ &= \frac{1}{2}(\theta_2-\theta_1) - \frac{1}{4}\left(\sin(2\theta_2)-\sin(2\theta_1)\right). \end{aligned}$$

By symmetry, B is given by the same expression, with θ_1 and θ_2 replaced by $\pi/2-\theta_2$ and $\pi/2-\theta_1$, respectively. Hence,

$$\begin{aligned} A + B = &\frac{1}{2}\left(\theta_2-\theta_1+(\pi/2-\theta_1)-(\pi/2-\theta_2)\right) \\ &- \frac{1}{4}\left(\sin(2\theta_2)-\sin(2\theta_1)\right) \\ &- \frac{1}{4}\left(\sin(2(\pi/2-\theta_1)-\sin(2(\pi/2-\theta_2)\right). \\ = &\;\theta_2-\theta_1, \end{aligned}$$

since $\sin(2(\pi/2-x)) = \sin(\pi-2x) = \sin(2x)$. Hence $A + B$ is equal to $\theta_2-\theta_1$, the length of the given arc, as desired.

4. A polynomial $P(x)$ is known to be of the form

$$P(x) = x^{15} - 9x^{14} + \cdots - 7.$$

where the ellipsis $(\cdots)$ represents unknown intermediate terms. It is also known that all roots of $P(x)$ are integers. Find the roots of $P(x)$.

Solution. Since $P(x)$ has degree 15, it has 15 roots (counted with multiplicity). Let $r_1, r_2, \dots, r_{15}$ denote these roots, which, by assumption, are all integers. Since $P(x)$ has leading term 1, it can be written as

$$P(x) = \prod_{i=1}^{15}(x-r_i).$$

Expanding this product we obtain

$$P(x) = x^{15} + \left(\sum_{i=1}^{15}(-r_i)\right)x^{14} + \cdots + \prod_{i=1}^{15}(-r_i).$$

Comparing this expression with the given form of $P(x)$, we get

$$(1)\quad \prod_{i=1}^{15} r_i = 7, \qquad (2)\quad \sum_{i=1}^{15} r_i = 9,$$

Equation (1) forces one of the roots r_i to be 7 or -7, and the remaining 14 roots to be 1 or -1. However, in the case when one of the roots is -7 the sum of all roots can be at most $-7 + 14 = 7$, contradicting equation (2). Hence one root must be 7, and the other 14 roots must be 1 or -1. Let n denote the number of roots 1, and $14 - n$ denote the number of roots -1. Then, (2) becomes

$$7 + n \cdot 1 + (14 - n) \cdot (-1) = 9,$$

or, equivalently, $-7 + 2n = 9$, which implies $n = 8$. Thus, the roots of the given polynomial, with multiplicities, are

$$7, \underbrace{1, \dots, 1}_{8}, \underbrace{-1, \dots, -1}_{6}.$$

5. Prove that the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{n^{2009}}{2^n}$$

is an integer.

Solution. For $k = 1, 2, \dots$, let

$$S_k(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)x^n, \quad T_k(x) = \sum_{n=1}^{\infty} n^k x^n,$$

and set

$$S_0(x) = T_0(x) = \sum_{n=0}^{\infty} x^n.$$

Thus, the given sum is $T_{2009}(1/2)$. To show that this is an integer, we will prove, more generally, that the numbers $S_k(1/2)$ and $T_k(1/2)$ are all integers.

We first note that, $S_0(x)(= T_0(x))$ is a geometric series with sum $(1 - x)^{-1}$. Differentiating this series k times (where k is a positive integer), we get

$$k!(1-x)^{-k-1} = S_0^{(k)}(x) == \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)x^{n-k} = x^{-k} S_k(x).$$

Thus,

$$S_k(1/2) = k!(1 - 1/2)^{-k-1}(1/2)^k = 2k!$$

for each $k = 0, 1, 2, \dots$. This shows that $S_k(1/2)$ is an integer for for each such k.

To show that the same is true for $T_k(1/2)$, it suffices to show that the function $T_k(x)$ can be expressed as a linear combination of the functions $S_i(x)$, $i = 0, 1, \dots, k$, with integer coefficients. This will follow if we can express each power t^k as linear combinations of terms $t^{\underline{i}} = t(t-1)\dots(t-i-1)$, $i = 0, 1, \dots, k$. (where $t^{\underline{0}} = 1$), with integer coefficients. This is easily proved by induction: For $k = 0$, the assertion holds since $t^0 = 1$ and $t^{\underline{0}} = 1$. Let $k \geq 1$ and assume that each power t^i, $i = 0, 1, \dots, k-1$ is a linear combination of $t^{\underline{j}}$'s with integer coefficients. Then, writing

$$t^{\underline{k}} = t(t-1)\dots(t-k+1) = t^k + \sum_{i=0}^{k-1} a_{i,k} t^i,$$

with integer coefficients $a_{i,k}$ and applying the induction hypothesis to the terms t^i in the latter sum, we see that t^k is a linear combination of terms $t^{\underline{i}}$, $i = 0, 1, \dots, k$, with integer coefficients, completing the induction.

6. Let $f(n)$ be a nonnegative real-valued function defined on all nonnegative integers and satisfying

(1) $$f(n+m) \leq f(n) + f(m) \quad (n, m \geq 0).$$

Prove that $f(n)/n$ converges as $n \to \infty$ and

(2) $$\lim_{n\to\infty} \frac{f(n)}{n} = \inf_{n\geq 1} \frac{f(n)}{n}.$$

Solution. First fix a positive integer k. Applying (1) with $m = k$ and $n = k, 2k, 3k, \dots$, we obtain

$$\begin{aligned} f(2k) &= f(k+k) \leq f(k) + f(k) = 2f(k), \\ f(3k) &= f(2k+k) \leq f(2k) + f(k) \leq 2f(k) + f(k) = 3f(k), \\ f(4k) &= f(3k+k) \leq f(3k) + f(k) \leq 3f(k) + f(k) = 4f(k), \qquad \dots \end{aligned}$$

and by induction we get

(3) $$f(hk) \leq hf(k)$$

for any positive integer h.

Next, let n be an arbitrary integer $> k$, and write n as $n = hk + r$, where $0 \leq r \leq h-1$ and $h = \lfloor n/k \rfloor$. Then, by (1) and (3),

$$f(n) = f(hk+r) \leq f(hk) + f(r) \leq \lfloor n/k \rfloor f(k) + f(r).$$

Dividing by n and letting $n \to \infty$, we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{f(n)}{n} &\leq \limsup_{n\to\infty} \frac{\lfloor n/k \rfloor}{n} f(k) + \limsup_{n\to\infty} \frac{\max_{r=0,1,\dots,k-1} f(r)}{n} \\ &= \frac{f(k)}{k} + 0. \end{aligned}$$

This inequality is true for any positive integer k. Since the left-hand side is independent of k and f is nonnegative, it follows that

$$\begin{aligned} 0 \le \limsup_{n\to\infty} \frac{f(n)}{n} &\le \inf_{k\ge 1} \frac{f(k)}{k} \\ &\le \liminf_{k\to\infty} \frac{f(k)}{k}. \le \limsup_{k\to\infty} \frac{f(k)}{k}. \end{aligned}$$

But this implies that the limit $\lim_{n\to\infty} f(n)/n$ exists, is finite, and is equal to $\inf_{k\ge 1} f(k)/k$, as claimed.