U OF I UNDERGRADUATE MATH CONTEST
MARCH 7, 2009

1. Given a nonnegative integer n, let $\overleftarrow{n}$ denote the integer obtained by reversing the digits of n in the standard decimal representation; for example, $\overleftarrow{935} = 539$. Let $f(n) = n + \overleftarrow{n}$, $g(n) = n - \overleftarrow{n}$, and $h(n) = f(g(n))$. For example, if $n = 935$, then $g(n) = 935 - 539 = 396$, and $h(n) = f(396) = 396 + 693 = 1089$; if $n = 701$, then $g(n) = 701 - 107 = 594$, $h(n) = 594 + 495 = 1089$.

   Prove that these results are no accident by showing that $h(n) = 1089$ for all three digit integers n whose first digit exceeds the last digit by at least 2.

2. Let S be a set of 16 distinct positive integers, all less than 60. Show that there exist four pairwise distinct elements $a, b, c, d \in S$ such that $a + b = c + d$.

3. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and B the area of the region lying to the right of the y-axis and to the left of s. Prove that the sum of these areas, $A + B$, depends only on the arc length, and not on the position, of s.

4. A polynomial $P(x)$ is known to be of the form

   $$P(x) = x^{15} - 9x^{14} + \cdots - 7.$$

   where the ellipsis $(\cdots)$ represents unknown intermediate terms. It is also known that all roots of $P(x)$ are integers. Find the roots of $P(x)$.

5. Prove that the sum of the infinite series

   $$\sum_{n=1}^{\infty} \frac{n^{2009}}{2^n}$$

   is an integer.

6. Let $f(n)$ be a nonnegative real-valued function defined on all nonnegative integers and satisfying

   (1) $$f(n+m) \le f(n) + f(m) \quad (n, m \ge 0).$$

   Prove that $f(n)/n$ converges as $n \to \infty$ and

   (2) $$\lim_{n\to\infty} \frac{f(n)}{n} = \inf_{n\ge 1} \frac{f(n)}{n}.$$

[Solutions at http://www.math.uiuc.edu/contests.html]