Suppose \( a \) and \( b \) are non-zero real numbers with \( a + b = 1/a + 1/b \). Show that \( a^3 + b^3 = 1/a^3 + 1/b^3 \).

**Solution.** For any fixed \( a \neq 0 \), consider the equation \( 1) a + x = 1/a + 1/x \). Multiplying through by \( x \) and solving the resulting quadratic equation \( x^2 + (a - 1/a)x - 1 = 0 \) for \( x \), one finds that \( x = -a \) and \( x = 1/a \) are the only solutions to (1). Hence the given equation \( a + b = 1/a + 1/b \) implies that \( b = -a \) or \( b = 1/a \). In either case, the asserted identity \( a^3 + b^3 = 1/a^3 + 1/b^3 \) holds.

**E2** Show that, for all positive integers \( n \), the number \( 3^{n+2} + 4^{2n+1} \) is divisible by 13.

**Solution.** Working modulo 13, we have

\[
3^{n+2} + 4^{2n+1} \equiv 9 \cdot 3^n + 4 \cdot 16^n \equiv 9 \cdot 3^n + 4 \cdot 3^n \equiv 13 \cdot 3^n \equiv 0 \mod 13.
\]

**E3** What is the 2000th digit in the sequence 12345678910111213...?

**Solution.** The sequence is generated by writing the sequence of positive integers down. The 9 single digit integers 1, 2, ..., 9 occupy positions 1 - 9 in the sequence; the 90 two digit integers 10, 11, ..., 99, occupy positions 10 - 189. To get to position 2000 requires 1811 additional digits. The sequence is generated by writing the sequence of positive integers down. The 9 single digit integers 1, 2, ..., 9 occupy positions 1 - 9 in the sequence; the 90 two digit integers 10, 11, ..., 99, occupy positions 10 - 189. To get to position 2000 requires 1811 additional digits.

Since the 603 three digit integers 100, 101, ..., 702 take up 1809 digits and occupy positions 190 - 1998, the next integer, 703, occupies positions 1999, 2000, and 2001. Hence the 2000th digit is 0.

**E4** How large can the product of a set of positive integers be if their sum is equal to 2000?

**Solution.** Suppose \( a_1, a_2, ..., a_n \) are positive integers with sum \( S = \sum_i a_i = 2000 \) for which the product \( P = \prod_i a_i \) is maximal. First note that if any \( a_i \) is \( \geq 4 \), then replacing \( a_i \) by the two integers 2 and \( a_i - 2 \) does not change \( S \), but increases, or leaves unchanged, \( P \), since \( 2(x - 2) = 2x - 4 \geq x \) for \( x \geq 4 \). We may therefore assume that \( a_i \leq 4 \) for all \( i \). Next, observe that at most one \( a_i \) can equal 1, since if there were \( k \geq 2 \) integers 1 among the \( a_i \), replacing these integers by the single integer \( k \) leaves \( S \) unchanged, but increases \( P \) by a factor \( k \). We may therefore assume that at most one of the \( a_i \) is equal to 1. Similarly, if there are at least 3 integers 2 among the \( a_i \), replacing three of these integers 2 by two integers 3 leaves again \( S \) unchanged, but increases \( P \) by a factor 9/8, and if there are at least 1 and one 2 among the \( a_i \), then replacing the pair (1, 2) by the single integer 3 leaves \( S \) unchanged, but increases \( P \) by a factor 3/2. Thus, there can be at most 2 numbers 2 among the \( a_i \), and if one of the \( a_i \) is equal to 1, the \( a_i \)'s cannot contain the number 2. Altogether we that the elements \( a_i \) of an extremal configuration consist of only the integers 1, 2, and 3, and that if \( n_k \) denotes the number of integers \( k \) among the \( a_i \) (for \( k = 1, 2, 3 \)), then \( (n_1, n_2) \) must be of one of the forms (1) \((0, 0)\), (2) \((1, 0)\), (3) \((0, 1)\), or (4) \((0, 2)\). The condition \( S = 2000 \) translates into \( n_1 + 2n_2 + 3n_3 = 2000 \), which implies \( n_1 + 2n_3 = 2000 \equiv 2 \mod 3 \). This condition is only satisfied in case (3). Hence the \( a_i \)'s consist of no 1, exactly one 2, and \( n_3 \) integers 3, where \( n_3 = (2000 - 2)/3 = 666 \). For this case, \( P = 2^{13} \cdot 666 \), which is the maximal product sought.

**E5** In a round-robin tournament with \( n \) players, \( P_1, P_2, ..., P_n \), each of the players plays a match against every other player. There are no ties, so each match ends in a win for one side and a loss for the other.
Show that, among any 10 consecutive integers, there is always one that has no common prime factor with any of the other integers.

Solution. Since the difference between any two integers in a set of $n$ consecutive integers is at most 9, the only possible common prime factors among two of these integers are 2, 3, 5, and 7. Thus, to prove the claim, it suffices to show that among any 10 consecutive integers there is at least one that is not divisible by any of these four prime factors. To see this, note first that any interval of length 10 contains (at least) three integers congruent to ±1 modulo 6, and none of these integers is divisible by 2 or 3. Of these integers, at most one is divisible by 5, and at most one is divisible by 7 (since the three integers are odd and fall in an interval of length 9, and odd multiples of 5 or of 7 are at least 10 (resp. 14) apart). Hence at least one integer is not divisible by any of the primes 2, 3, 5, and 7.

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Advanced Problems

A2 Evaluate the infinite product $\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$.

Solution. Write the $n$th factor as $a_n/b_n$, where $a_n = n^3 - 1$ and $b_n = n^3 + 1$, and let $P_N = \prod_{n=2}^{N} a_n/b_n$ be the $N$th partial product. Note that $a_n$ and $b_n$ factor into $a_n = (n - 1)(n^2 + n + 1)$ and $b_n = (n + 1)(n^2 - n + 1) = (n + 1)((n - 1)^2 + (n - 1) + 1)$, so

$$P_N = \prod_{n=2}^{N} \frac{(n - 1)(n^2 + n + 1)}{(n + 1)((n - 1)^2 + (n - 1) + 1)} = \frac{1 \cdot 2 \cdot (N^2 + N + 1)}{(1^2 + 1 + 1) \cdot N(N + 1)}.$$ 

As $N \to \infty$, this converges to 2/3, so the value of the given infinite product is $\lim_{N \to \infty} P_N = 2/3$.

A3 Show that, if $n$ is odd, then $1^n + 2^n + \cdots + n^n$ is divisible by $n^2$.

Solution. Since $n^n$ is divisible by $n^2$, it suffices to consider the sum $1^n + 2^n + \cdots + (n - 2)^n + (n - 1)^n$. Since $n$ is odd, we can match up $k^n$ with $(n - k)^n$, for $k = 1, 2, \ldots, (n - 1)/2$, and it suffices to show that $k^n + (n - k)^n$ is divisible by $n^2$ for each $k$. Expanding $(n - k)^n$ and reducing modulo $n^2$, we see that

$$(n - k)^n = (-k)^n + \binom{n}{1} n^1 (-k)^{n-1} + \sum_{i=2}^{n} \binom{n}{i} n^i (-k)^{n-i} \equiv (-1)^n k^n \mod n^2,$$

so $k^n + (n - k)^n \equiv (1 + (-1)^n) k^n \equiv 0$ modulo $n^2$, since $n$ is odd.

A4 (Corrected version) Evaluate the infinite series $\sum_{n=-\infty}^{\infty} (-1)^n x^{n(n+1)/2}$ for $|x| < 1$.

Solution. [On the original problem sheet, this problem was misstated, with coefficient $(-1)^{n(n+1)/2}$ in place of $(-1)^n$.]

The ratio test shows that the infinite series converges absolutely for $|x| < 1$, so the terms in this
series can be rearranged. Let \( a_n = x^{n(n+1)/2} \). Then \( a_{-n} = x^{(-n)(-n+1)/2} = x^{n(n-1)/2} = a_{n-1} \).

Hence, the given series is equal to

\[
\sum_{n=0}^{\infty} (-1)^n a_n + \sum_{n=1}^{\infty} (-1)^{-n} a_{-n} = \sum_{n=0}^{\infty} (-1)^n a_n + \sum_{n=1}^{\infty} (-1)^n a_{n-1} = \sum_{n=0}^{\infty} ((-1)^n + (-1)^{n+1}) a_n = 0.
\]

A5 Determine, with proof, all functions \( f \) defined on the set of integers and satisfying \( f(n + m) + f(n - m) = 2(f(m) + f(n)) \) for all \( n \) and \( m \).

**Solution.** Setting \( m = n = 0 \) gives \( 2f(0) = 4f(0) \) which implies \( f(0) = 0 \). Also, setting \( n = 0 \), we see that \( f(m) + f(-m) = 2(f(m) + f(0)) = 2f(m) \), which implies \( f(-m) = f(m) \) for all \( m \). Next, let \( \lambda = f(1) \), and apply the given equation with \( m = 1 \) to get \( f(n + 1) + f(n - 1) = 2(\lambda + f(n)) \) or, equivalently, \( f(n + 1) = 2f(n) - f(n - 1) + 2\lambda \) for all \( n \). This is a two-term recurrence relation for \( f(n) \), and by induction, using the values \( f(0) = 0 \) and \( f(1) = \lambda \) one can easily check that \( (3) \) implies \( (4) f(n) = \lambda n^2 \) for all positive integers \( n \). In view of \( (2) \), \( (4) \) holds for negative integers \( n \) as well. Thus, any solution to the given functional equation must be of the form \( (4) \). Conversely, it is easy to check that any function of this form satisfies the given functional equation.