2021 UI MOCK PUTNAM CONTEST
October 30, 2021, 1 pm – 4 pm
Solutions

1. Let m be an integer ≥ 2. Prove that any positive integer $n > m$ can be written in the form $n = a + b$ where a and b are positive integers such that (1) a divides m and (2) b is a coprime with m (i.e., has no common prime factor with m).

   Solution. Write $m = \prod_{i=1}^{\ell} p_i^{\alpha_i}$, let $a = \prod_{i=1}^{\ell} p_i^{\beta_i}$, where $\beta_i = 1$ if p_i does not divide n, and $\beta_i = 0$ if p_i divides n, and let $b = n - a$.

   Clearly, a divides m. We now show that b is coprime with m. Indeed, suppose there is a prime number p that divides both m and b. Since p divides m, $p = p_i$ for some i. If $\beta_i = 1$, then p_i divides a, and since, by assumption, p_i divides b, it follows that p_i also divides $n = a+b$. But this contradicts the definition of β_i. If $\beta_i = 0$, then p_i does not divide a, while, by assumption, p_i divides b. But then p_i does not divide $n = a + b$, and we again have a contradiction to the definition of β_i.

   Remark: Several students defined a as $a = m/d$, where d is the greatest common divisor of m and n. This would work in the case m/d and d are coprime, but in more general cases it does not work. For a counterexample, let $n = 6$ and $m = 4$. Then $d = 2$, $a = m/d = 2$, and so $b = n - a = 4$, which is not coprime with m.

2. Suppose $x_1, \ldots, x_n$ are real numbers, and $\sum_{i=1}^{n} x_i = 0$. Prove that there exists $k \in \{1, \ldots, n\}$ so that all sums x_k, $x_k + x_{k+1}$, ..., $x_k + \ldots + x_n$, $x_k + \ldots + x_n + x_1$, $x_k + \ldots + x_n + x_1 + \ldots x_{k-1}$ are non-negative.

   Solution. Extend the definition of x_i to all $i \geq 1$ by letting $x_{n+i} = x_i$. We then need to show that, for some k, the n sums $\sum_{i=k}^{k+h-1} x_i$, $h = 1, \ldots, n$, are all nonnegative.

   Consider the sums $S_{k,m} = \sum_{i=k}^{m-1} x_i$, where k and m are positive integers with $k < m$. Since $\sum_{i=1}^{n} x_i = 0$, there are only finitely many values for the sums $S_{k,m}$, so there exist k and $m > k$ such that $S_{k,m}$ is maximal. We claim that this k works.

   Indeed, suppose for the sake of contradiction that $S_{k,j} < 0$ for some $j \in \{k+1, \ldots, k+n\}$. If $j < m$, we have $S_{k,m} = S_{k,j} + S_{j,m} < S_{j,m}$, contradicting the maximality of $S_{k,m}$. A similar contradiction arises case of $j > m$.

3. Suppose n is an integer ≥ 2 and α a real number such that $\sin\alpha \neq 0$. Prove that the polynomial $P(x) = x^n \sin\alpha - x \sin n\alpha + \sin(n-1)\alpha$ is divisible by the polynomial $Q(x) = x^2 - 2x\cos\alpha + 1$.

   Solution. By the Factor Theorem, a polynomial of degree r can be factored as $c\prod_{i=1}^{r}(x - \omega_i)$, where c is a constant and the ω_i are the roots of the polynomial counted with multiplicity. Thus, to show that Q divides P, it suffices to show that every root of Q is also a root of P.

   By the quadratic formula, the roots of $Q(x)$ are $\cos\alpha \pm i\sin\alpha = e^{\pm i\alpha}$. Since P has real coefficients, we have $P(e^{-i\alpha}) = \overline{P(e^{i\alpha})}$, so it suffices to show that $e^{i\alpha}$ is a root of P. We have

   $$P(e^{i\alpha}) = e^{in\alpha}\frac{e^{i\alpha} - e^{-i\alpha}}{2i} - e^{i\alpha}\frac{e^{in\alpha} - e^{-in\alpha}}{2i} + \frac{e^{i(n-1)\alpha} - e^{-i(n-1)\alpha}}{2i} = 0,$$

   as desired.

4. Find, with proof, all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

   $$f(x+y) = f(x) + f(y) + 2xy \quad \text{for all } x, y \in \mathbb{R}.$$

Solution. We claim that the functions are exactly those of the form $(*)$ $f(x) = x^2 + dx$, where d is a constant.

For the proof, first set $x = y = 0$ in the given equation to obtain $f(0) = f(0) + f(0)$ and therefore $f(0) = 0$.

Next, consider y as fixed and take derivatives with respect to x in the given equation to get (1) $f'(x + y) = f'(x) + 2y$ and (2) $f''(x + y) = f''(x)$.

Setting $x = 0$ in (2) shows that $f''(y) = c$, where $c = f''(0)$ is a constant. Therefore (3) $f'(x) = cx+d$, where d is a constant. Substituting (3) into (1) gives $c(x + y) + d = cx + d + 2y$, so we must have $c = 2$ and therefore (4) $f'(x) = 2x + d$. Finally, integrating (4) and using the fact that $f(0) = 0$ gives $f(x) = x^2 + dx$. Thus, $f(x)$ must be of the form $(*)$.

Conversely, if $f(x) = x^2 + dx$, then

$$f(x + y) = (x + y)^2 + d(x + y) = x^2 + dx + y^2 + dy + 2xy = f(x) + f(y) + 2xy,$$

so any function f of the form $(*)$ satisfies the given equation.

5. Suppose $\cos(2\pi\alpha) = 1/7$. Prove that α is irrational.

Solution. Assume that $\alpha = m/n$ is rational. Then $e^{2\pi i\alpha n} = e^{2\pi i m} = 1$. Applying Euler's formula and the Binomial Theorem we obtain

$$1 = e^{2\pi i\alpha n} = (\cos 2\pi\alpha + i \sin 2\pi\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} i^k (\sin 2\pi\alpha)^k (\cos 2\pi\alpha)^{n-k}.$$

Taking the real part on each side eliminates those terms on the right in which k is odd, so we obtain

$$\begin{aligned} 1 &= \sum_{h=0}^{\lfloor n/2\rfloor} \binom{n}{2h} (-1)^h (\sin 2\pi\alpha)^{2h} (\cos 2\pi\alpha)^{n-2h} \\ &= \sum_{h=0}^{\lfloor n/2\rfloor} \binom{n}{2h} (-1)^h (1 - \cos^2 2\pi\alpha)^h (\cos 2\pi\alpha)^{n-2h} + R \\ &= (\cos 2\pi\alpha)^n \sum_{h=0}^{\lfloor n/2\rfloor} \binom{n}{2h} + R, \end{aligned}$$

where R is a polynomial in $\cos 2\pi\alpha$ with integer coefficients of degree at most $n - 2$. Since

$$\sum_{h=0}^{\lfloor n/2\rfloor} \binom{n}{2h} = \frac{1}{2} \sum_{k=0}^{2n} \binom{n}{k} (1 + (-1)^k) = \frac{1}{2} \left((1 + 1)^n + (1 - 1)^n\right) = 2^{n-1},$$

we get

$$1 = (\cos 2\pi\alpha)^n 2^{n-1} + R = (1/7)^n 2^{n-1} + (1/7)^{n-2} q,$$

for some integer q. We therefore obtain $7^n = 2^{n-1} + 7^2 q$. This is a contradiction since the left side is divisible by 7 while the right side is not.

6. Suppose a, b, c, d, n are integers so that 5 does not divide d, but divides $an^3 + bn^2 + cn + d$. Prove that there exists an integer m so that 5 divides $dm^3 + cm^2 + bm + a$.

Solution. Since $an^3 + bn^2 + cn + d$ is divisible by 5, while d is not divisible by 5, n cannot be divisible by 5. Since 5 is prime, n has a modular inverse modulo 5, i.e., there exists an integer $m \in \{1, \dots, 4\}$ such that $mn \equiv 1 \bmod 5$. We claim that this m has the desired property. Indeed, multiplying the congruence $an^3 + bn^2 + cn + d \equiv 0 \bmod 5$ by m^3 gives

$$\begin{aligned} 0 &\equiv an^3m^3 + bn^2m^3 + cnm^3 + dm^3 \\ &= a(mn)^3 + b(mn)^2 m + c(mn)m^2 + dm^3 \\ &\equiv a + bm + cm^2 + dm^3 \bmod 5, \end{aligned}$$

so 5 divides $a + bm + cm^2 + dm^3$ as claimed.

7. Suppose $A_1, \dots, A_m$ are proper subsets of $\{1, \dots, n\}$ ($n \geq 3$) with the property that, for any distinct $i, j \in \{1, \dots, n\}$, there exists a unique $k \in \{1, \dots, m\}$, so that $i, j \in A_k$. Prove that $m \geq n$.

Solution. Suppose, for the sake of contradiction, that $n > m$.
For $i \in \{1, \dots, n\}$ denote by $f(i)$ the cardinality of $\{k : i \in A_k\}$.
Note that, if $i \notin A_s$, then $f(i) \geq |A_s|$. Indeed, for any $j \in A_s$, there exists k_j s.t. $i, j \in A_{k_j}$; all the indices k_j must be different.
By our assumption, $n > m$, hence $nf(i) > m|A_s|$, hence $n(m - f(i)) < m(n - |A_s|)$. Then

$$1 = \sum_{i=1}^{n} \frac{m - f(i)}{n(m - f(i))} = \sum_{i=1}^{n} \sum_{k: i \notin A_k} \frac{1}{n(m - f(i))} > \sum_{k=1}^{m} \sum_{i \notin A_k} \frac{1}{m(n - |A_k|)}$$
$$= \frac{1}{m} \sum_{k=1}^{m} \sum_{i \notin A_k} \frac{1}{n - |A_k|} = \frac{1}{m} \sum_{k=1}^{m} \frac{n - |A_k|}{n - |A_k|} = 1,$$

which is impossible.