2021 UI MOCK PUTNAM CONTEST
October 30, 2021, 1 pm – 4 pm

1. Let m be an integer ≥ 2. Prove that any positive integer $n > m$ can be written in the form $n = a + b$ where a and b are positive integers such that (1) a divides m and (2) b is a coprime with m (i.e., has no common prime factor with m).

2. Suppose $x_1, \ldots, x_n$ are real numbers, and $\sum_{i=1}^{n} x_i = 0$. Prove that there exists $k \in \{1, \ldots, n\}$ so that all sums x_k, $x_k + x_{k+1}$, ..., $x_k + \ldots + x_n$, $x_k + \ldots + x_n + x_1$, $x_k + \ldots + x_n + x_1 + \ldots x_{k-1}$ are non-negative.

3. Suppose n is an integer ≥ 2 and α a real number such that $\sin \alpha \neq 0$. Prove that the polynomial $P(x) = x^n \sin \alpha - x \sin n\alpha + \sin(n-1)\alpha$ is divisible by the polynomial $Q(x) = x^2 - 2x \cos \alpha + 1$.

4. Find, with proof, all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$f(x + y) = f(x) + f(y) + 2xy \quad \text{for all } x, y \in \mathbb{R}.$$

5. Suppose $\cos(2\pi\alpha) = 1/7$. Prove that α is irrational.

6. Suppose a, b, c, d, n are integers so that 5 does not divide d, but divides $an^3 + bn^2 + cn + d$. Prove that there exists an integer m so that 5 divides $dm^3 + cm^2 + bm + a$.

7. Suppose $A_1, \ldots, A_m$ are proper subsets of $\{1, \ldots, n\}$ ($n \geq 3$) with the property that, for any distinct $i, j \in \{1, \ldots, n\}$, there exists a unique $k \in \{1, \ldots, m\}$, so that $i, j \in A_k$. Prove that $m \geq n$.