2020 UI MOCK PUTNAM PROBLEMS
Solutions

1. Given an arbitrary positive integer N, show that the sum

$$\sum_{n=N+1}^{2N} \sqrt{n^2+1}$$

is **not** an integer.

Solution. Using the inequality

$$n < \sqrt{n^2+1} = n\sqrt{1+\frac{1}{n^2}} < n\left(1+\frac{1}{n^2}\right) = n + \frac{1}{n}.$$

we get, for any positive integer N,

$$\sum_{n=N+1}^{2N} n < S(N) < \sum_{n=N+1}^{2N} n + \sum_{n=N+1}^{2N} \frac{1}{n} \le \sum_{n=N+1}^{2N} n + \sum_{n=N+1}^{2N} \frac{1}{N+1} < \sum_{n=N+1}^{2N} n + 1.$$

Thus $S(N)$ lies strictly between two consecutive integers and therefore cannot be an integer.

2. Suppose $\mathcal{P}$ is a finite set of points in the plane such that the mutual distances between these points are all distinct. For each point P in $\mathcal{P}$ define its *closest neighbor* to be the point Q in $\mathcal{P}$ whose distance to P is minimal. (The assumption that the mutual distances are distinct ensures that each point P has a unique closest neighbor Q.) Prove that there does not exist a point Q that is the closest neighbor to more than 5 points.

Solution. Suppose, to get a contradiction, that there is a point Q that is closest neighbor to 6 points in $\mathcal{P}$, say $P_1, \dots, P_6$. By the pigeonhole principle, for two of these points, say P_i and P_j, the angle at Q of the triangle P_iQP_j must be $\le \pi/3$. But then at least one of the other two angles of this triangle, say the angle at P_i, must be $\ge \pi/3$. By the sine law, it follows that $|P_jQ| \ge |P_jP_i|$, and since the pairwise distances between the points are distinct, we must have $|P_jQ| > |P_jPi|$. But this contradicts the assumption that Q is the point closest to P_j.

3. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} \binom{4n+2}{4k} = 2^{4n}.$$

Solution. Using the recurrence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ and the symmetry property $\binom{n}{k} = \binom{n}{n-k}$, the sum on the left can be written as

$$\begin{aligned}
\sum_{k=0}^{n} \binom{4n+2}{4k} &= 1 + \sum_{k=1}^{n} \left(\binom{4n+1}{4k-1} + \binom{4m+1}{4k} \right) \\
&= 1 + \sum_{k=1}^{n} \left(\binom{4n}{4k-2} + \binom{4n}{4k-1} + \binom{4n}{4k-1} + \binom{4n}{4k} \right) \\
&= 1 + \sum_{k=1}^{n} \binom{4n}{4k-2} + \sum_{k=1}^{n} \binom{4n}{4k-1} + \sum_{k=1}^{n} \binom{4n}{4m-4k+1} + \sum_{k=1}^{n} \binom{4n}{4k} \\
&= 1 + \sum_{k=1}^{n} \binom{4n}{4k-2} + \sum_{k=1}^{n} \binom{4n}{4k-1} + \sum_{h=1}^{n} \binom{4n}{4h-3} + \sum_{k=1}^{\infty} \binom{4n}{4k} \\
&= \sum_{m=0}^{4n} \binom{4n}{m} = 2^{4n}
\end{aligned}$$

4. Suppose $P(x)$ is a polynomial with real coefficients of degree 2020 such that the values $P(0)$, $P(1)$,..., $P(2020)$ are all integers. Prove that $P(2021)$ is also an integer.

Solution. More generally, we will show that if $P(x)$ is a polynomial of degree $n \geq 0$ such that $P(0), P(1), \ldots, P(n)$ are integers, then $P(n+1)$ is also an integer.

We use induction on n. In the base case $n = 0$, $P(x)$ is a constant polynomial, so the statement holds trivially.

For the induction step, let $k \geq 1$ be given suppose our statement has been established for $n = k-1$. Suppose that P is a polynomial of degree k such that $P(i)$ is an integer for $i = 0, 1, \ldots, k$. Let $Q(x) = P(x+1) - P(x)$. Then $Q(x)$ is a polynomial of degree $k-1$ such that $Q(i) = P(i+1) - P(i)$ is an integer for $i = 0, 1, \ldots, k-1$. By the induction hypothesis, it follows that $Q(k)$ is also an integer. Therefore $P(k+1) = P(k) + Q(k)$ is an integer as well, and the induction is complete.

5. Suppose $f(x)$ is a continuous function on the interval $[0, 1]$ satisfying

$$\int_0^1 x^n f(x)\, dx = 1 \quad \text{and} \quad \int_0^1 x^k f(x)\, dx = 0 \quad \text{for } k = 0, 1, \ldots, n-1.$$

Prove that

$$\max_{0 \leq x \leq 1} |f(x)| \geq (n+1)2^n.$$

Solution. (Putnam 1972, Problem A6) We argue by contradiction. Suppose f satisfies the conditions of the problem, but $\max_{0 \leq x \leq 1} |f(x)| < (n+1)2^n$.

Let $I = \int_0^1 (x - 1/2)^n f(x)$. Expanding $(x - 1/2)^n$ by the binomial theorem, we get

$$I = \int_0^1 \sum_{k=0}^{n} \binom{n}{k} x^k (-1/2)^{n-k} f(x) dx = \sum_{k=0}^{n} \binom{n}{k} (-1/2)^{n-k} \int_0^1 x^k f(x) dx = \int_0^1 x^n f(x) dx = 1.$$

On the other hand, by our assumption on the maximum value of $|f(x)|$, we have

$$I < (n+1)2^n \int_0^1 |x - 1/2|^n\, dx = (n+1)2^n \cdot 2 \int_0^{1/2} y^n\, dx = (n+1)2^n \cdot 2 \frac{(1/2)^{n+1}}{n+1} = 1,$$

which is a contradiction.

6. Let $p_1, \ldots, p_k$ be distinct primes, and suppose $P(x)$ is a polynomial with integer coefficients such that, for every natural number n, $P(n)$ is divisible by (at least) one of the primes $p_1, \ldots, p_k$. Prove that there exists a prime p_i, $1 \leq i \leq k$, such that, for all positive integers n, $P(n)$ is divisible by p_i.

Solution. We argue by contradiction. Suppose that for each $i = 1, \ldots, k$ there exists a positive integer n_i such that $P(n_i)$ is **not** divisible by p_i, i.e., such that $P(n_i) \not\equiv 0 \bmod p_i$. By the Chinese Remainder Theorem there exists a positive integer N such that $N \equiv n_i \bmod p_i$ for each i. Then $P(N) \equiv P(n_i) \bmod p_i$ for each i. But since $P(n_i) \not\equiv 0 \bmod p_i$, it follows that $P(N) \not\equiv 0 \bmod p_i$ for each i. Thus, $P(N)$ is not divisible by any prime p_i, a contradiction.

7. Let A be the set of positive integers whose decimal representation does not contain the digit 0. Determine, with proof, the set of all positive real numbers p for which the series $\sum_{n \in A} \frac{1}{n^p}$ converges.

Solution. We will show that the series converges if and only if $p > \log 9/\log 10$.

Let $S = \sum_{n\in A} 1/n^p$ be the given series. We will split the sum S into intervals $I_k = [10^{k-1}, 10^k)$. Note that the integers in A that fall into I_k are exactly the integers with k decimal digits, none of which is 0. Thus $A \cap I_k$ contains 9^k integers. It follows that

$$\frac{9^k}{10^{kp}} \le \sum_{n\in A\cap I_k} \frac{1}{n^p} \le \frac{9^k}{10^{(k-1)p}},$$

and hence

$$\sum_{k=1}^{\infty} \frac{9^k}{10^{kp}} \le S \le \sum_{k=1}^{\infty} \frac{9^k}{10^{(k-1)p}} = 10^p \sum_{k=1}^{\infty} \frac{9^k}{10^{kp}}$$

Thus, S converges if and only if the series $\sum_{k=1}^{\infty} 9^k/10^{kp}$ converges. The latter series is a geometric series with ratio $r = 9/10^p$ and thus converges if and only if $9/10^p < 1$, i.e., if and only if $p > (\log 9)/(\log 10)$.