2020 UI MOCK PUTNAM PROBLEMS

1. Given an arbitrary positive integer N, show that the sum

$$\sum_{n=N+1}^{2N} \sqrt{n^2+1}$$

is **not** an integer.

2. Suppose $\mathcal{P}$ is a finite set of points in the plane such that the mutual distances between these points are all distinct. For each point P in $\mathcal{P}$ define its *closest neighbor* to be the point Q in $\mathcal{P}$ whose distance to P is minimal. (The assumption that the mutual distances are distinct ensures that each point P has a unique closest neighbor Q.) Prove that there does not exist a point Q that is the closest neighbor to more than 5 points.

3. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} \binom{4n+2}{4k} = 2^{4n}.$$

4. Suppose $P(x)$ is a polynomial with real coefficients of degree 2020 such that the values $P(0)$, $P(1)$,..., $P(2020)$ are all integers. Prove that $P(2021)$ is also an integer.

5. Suppose $f(x)$ is a continuous function on the interval $[0,1]$ satisfying

$$\int_0^1 x^n f(x)\,dx = 1 \quad \text{and} \quad \int_0^1 x^k f(x)\,dx = 0 \quad \text{for } k = 0, 1, \ldots, n-1.$$

Prove that

$$\max_{0\le x\le 1} |f(x)| \ge (n+1)2^n.$$

6. Let $p_1, \ldots, p_k$ be distinct primes, and suppose $P(x)$ is a polynomial with integer coefficients such that, for every natural number n, $P(n)$ is divisible by (at least) one of the primes $p_1, \ldots, p_k$. Prove that there exists a prime p_i, $1 \le i \le k$, such that, for all positive integers n, $P(n)$ is divisible by p_i.

7. Let A be the set of positive integers whose decimal representation does not contain the digit 0. Determine, with proof, the set of all positive real numbers p for which the series $\sum_{n\in A} \frac{1}{n^p}$ converges.