Elementary Problems

E1 Evaluate \( f(n) = 1^2 - 2^2 + 3^2 - \cdots + (2n - 1)^2 - (2n)^2. \)

E2 Show that, if \( n \) is odd, then \( 1^n + 2^n + \cdots + n^n \) is divisible by \( n^2 \).

E3 Let \( a_1 = 1, a_2 = 1, a_3 = -1 \), and for \( n > 3 \) define \( a_n \) by \( a_n = a_{n-1}a_{n-3} \). Find \( a_{2001} \).

E4 Evaluate the sum \( \sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k \).

Advanced Problems

A1 Let \( n \) and \( m \) be positive integers with \( n \geq 2m \). How many binary strings of length \( n \) are there that contain exactly \( m \) blocks of the form 01?

A2 Let \( H_n = \sum_{k=1}^{n} \frac{1}{k} \). Show that \( \lim_{n \to \infty} (H_n - \log n) = 1 - \int_0^1 \{ \frac{1}{x} \} \, dx \), where \( \{ y \} \) denotes the fractional part of \( y \).

A3 Let \( a_1 = \sqrt{2} \), and for \( n > 1 \) define \( a_n \) by \( a_n = (\sqrt{2})^{a_{n-1}} \). Prove that the sequence \( \{a_n\} \) converges and determine its limit.

A4 Find all polynomials \( P(x) \), all of whose roots are real and which satisfy (*) \( P(x^2 - 1) = P(x)P(-x) \) for all \( x \).