2019 UI MOCK PUTNAM CONTEST
October 12, 2019, 1 pm – 4 pm
Solutions

1. Prove that, for any real numbers $a < b$,

$$\int_a^b (x^2+1)e^{-x^2}dx \ge e^{-a^2} - e^{-b^2}.$$

Solution. (IMC 2010)
We have

$$e^{-a^2} - e^{-b^2} = \int_a^b 2xe^{-x^2}dx \le \int_a^b (1+x^2)e^{-x^2}dx,$$

since $2x = (1+x^2) - (1-x)^2 \le 1 + x^2$ for any x.

2. Given two nonnegative integers a and b, say that a **dominates** b if each digit in the binary expansion of a is greater or equal to the corresponding digit in the binary expansion of b. For example, the number $5 = (101)_2$ dominates the numbers $4 = (100)_2$, $1 = (1)_2$, and $0 = (0)_2$, but it does not dominate the numbers $3 = (11)_2$ or $2 = (10)_2$.

Find, with proof, a simple general formula for the number of pairs (a, b) of integers in the interval $[0, 2^n)$ such that a dominates b.

Solution. Answer: $\boxed{3^n}$
Proof: The nonnegative integers in $[0, 2^n)$ are the integers of the form $\sum_{i\in I} 2^i$, where I is a subset $\{1, \ldots, n\}$. Moreover, if $a = \sum_{i\in I} 2^i$ and $b = \sum_{i\in J} 2^i$ are two such integers, then a dominates b if and only if $J \subseteq I$. Thus, the number of pairs of integers (a, b) in $[0, 2^n)$ with a dominating b is equal to the number of pairs (I, J), where I, J are subsets of $\{1, \ldots, n\}$ and $J \subseteq I$. Now, for a given subset I with k elements there are exactly 2^k subsets $J \subseteq I$. Since there are $\binom{n}{k}$ subsets of $\{1, \ldots, n\}$ with k elements, the total number of pairs (I, J) of the desired form is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n = 3^n,$$

by the Binomial Theorem.

3. Prove that, for $0 < x < \pi/4$,

$$(\cos x)^{\cos^2 x} > (\sin x)^{\sin^2 x}.$$

Solution. Setting $t = \cos^2 x$ and squaring each side we see that the stated inequality is equivalent to the inequality $(*)$ $t^t > (1-t)^{1-t}$ for $1/2 < t < 1$. Further, taking logarithms, we see that $(*)$ is equivalent to $(**)$ $f(t) > 0$ for $1/2 < t < 1$, where $f(t) = t\log t - (1-t)\log(1-t)$.
We have $f(1/2) = 0$, and (e.g., by L'Hopital's Rule)

$$\lim_{t\to 1^-} f(t) = \lim_{t\to 1^-} t\log t - \lim_{t\to 1^-} (1-t)\log(1-t) = 0.$$

Now consider the derivative of f, $f'(t) = 1 + \log t + \log(1-t) + 1 = \log(e^2 t(1-t))$. This function is continuous on the interval $[1/2, 1)$, it is equal to $\ln(e^2/4) > 0$ at $t = 1/2$, it is decreasing for $1/2 < t < 1$, and it approaches $-\infty$ as $t \to 1^-$. Thus there exists a unique value $t_0 \in (1/2, 1)$ such that $f'(t) > 0$ for $1/2 < t < t_0$ and $f'(t) < 0$ for $t_0 < t < 1$. Hence we have $f(t) > f(1/2) = 0$ for $1/2 < t \le t_0$ and $f(t) > \lim_{s\to 1^-} f(s) = 0$ for $t_0 \le t < 1$. This proves $(**)$.

4. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \ldots$. Evaluate

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!}.$$

Solution. (B5, Putnam 1975) Answer: $\boxed{e^e}$
Proof. We first show by induction that

$$f_n(x) = \sum_{k=0}^{\infty} \frac{k^n x^k}{k!} \quad (x \in \mathbb{R}) \tag{1}$$

for $n = 0, 1, \ldots$. The base case $n = 0$ follows from the Taylor series expansion $e^x = \sum_{k=0}^{\infty} x^k/k!$. Now let $n \geq 0$ be given and suppose (1) holds for this value of n. Differentiating termwise we get

$$f_{n+1}(x) = xf_n'(x) = x \sum_{k=0}^{\infty} \frac{k^n \cdot kx^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{k^{n+1} x^k}{k!},$$

which shows that (1) holds for the case $n+1$ and completes the induction.
Using (1) we get

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{k^n 1^k}{k! n!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{k^n}{n!} = \sum_{k=0}^{\infty} \frac{e^k}{k!} = e^e.$$

5. Consider a sequence of coins $C_1, C_2, \ldots$ such that coin C_n comes up heads with probability $1/n$. Let p_n be the probability of getting an *even* number of heads if coin C_n is flipped n times. Determine, with proof, $\lim_{n\to\infty} p_n$.

Solution. Answer: $\boxed{(1/2)(1 + e^{-2})}$
Proof: Let X_n denote the number of heads obtained in n flips with coin C_n. By the binomial distribution we have

$$\begin{aligned}
P(X_n \text{ is even}) &= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} \frac{1}{2}\left(1 + (-1)^k\right) \\
&= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} + \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k}.
\end{aligned}$$

Applying the Binomial Theorem, this becomes

$$\frac{1}{2}\left(\frac{1}{n} + \left(1 - \frac{1}{n}\right)\right)^n + \frac{1}{2}\left(\frac{-1}{n} + \left(1 - \frac{1}{n}\right)\right)^n = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{2}{n}\right)^n.$$

Since, for any real number x, $\lim_{n\to\infty}(1 + x/n)^n = e^x$, it follows that the latter expression converges to $(1/2)(1 + e^{-2})$ as $n \to \infty$.

6. Determine, with proof, the set of all pairs (α, β) of positive real numbers for which the series

$$\sum_{m,n \in \mathbb{N}} \frac{(mn)^\alpha}{(m+n)^\beta}$$

converges.

Solution. We will show that the series converges if and only if $\boxed{\beta > 2\alpha + 2}$.
Proof: We first show that if $(*)$ $\beta > 2\alpha + 2$, then the series converges. We have

$$\sum_{m,n\in\mathbb{N}} \frac{(mn)^\alpha}{(m+n)^\beta} \le 2\sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(mn)^\alpha}{(m+n)^\beta}$$
$$\le 2\sum_{n=1}^{\infty} n\cdot\frac{n^{2\alpha}}{n^\beta} = 2\sum_{n=1}^{\infty} n^{2\alpha+1-\beta} < \infty,$$

since, by our assumption $(*)$, the exponent of n in the last series is < -1. Hence the given series converges when $(*)$ holds.
Next we show that if $(**)$ $\beta \le 2\alpha+2$, then the given series diverges. To this end consider those pairs (m,n) satisfying $m \le n < 2m$. For each such pair we have

$$\frac{(mn)^\alpha}{(m+n)^\beta} \ge \frac{m^{2\alpha}}{(3m)^\beta} = \frac{m^{2\alpha-\beta}}{3^\beta}.$$

Since, for each fixed m, there are exactly m values of n with $m \le n < 2m$, it follows that

$$\sum_{m,n\in\mathbb{N}} \frac{(mn)^\alpha}{(m+n)^\beta} \ge 3^{-\beta}\sum_{m=1}^{\infty} m^{2\alpha-\beta+1} \ge 3^{-\beta}\sum_{m=1}^{\infty} m^{-1} = \infty.$$

Hence the given series diverges under condition $(**)$. This completes the proof.

7. Find, with proof, all integrable functions $g : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^1 g(f(x))\,dx = g\left(\int_0^1 f(x)\,dx\right)$$

for every integrable function $f : [0,1] \to \mathbb{R}$.

Solution. We will show that the functions $g(x)$ satisfying the given functional equation are exactly the linear functions, i.e., the functions of the form $g(x) = \alpha x + \beta$ for some real numbers α and β.
First observe that, if $g(x) = \alpha x + \beta$, then

$$\int_0^1 g(f(x))\,dx = \int_0^1 (\alpha f(x) + \beta)dx = \alpha\int_0^1 f(x)dx + \beta\int_0^1 1\,dx = g\left(\int_0^1 f(x)\,dx\right).$$

Thus, any such function $g(x)$ satisfies the given functional equation.
Now let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying the given functional equation. Given real numbers $a < b$ and $t \in [0,1]$, consider the function

$$f(x) = \begin{cases} a & \text{if } 0 \le x \le t, \\ b & \text{if } t < x \le 1. \end{cases}$$

Then $\int_0^1 f(x)dx = ta + (1-t)b$, and therefore

$$g\left(\int_0^1 f(x)dx\right) = g(ta + (1-t)b). \tag{1}$$

On the other hand, we have

$$\int_0^1 g(f(x))\,dx = \int_0^t g(a)dx + \int_t^1 g(b)dx = tg(a) + (1-t)g(b). \tag{2}$$

By our assumption, the expressions (1) and (2) must be equal, so g must satisfy the equation

$$g(ta + (1-t)b) = tg(a) + (1-t)g(b) \quad (0 \le t \le 1). \tag{1}$$

But this implies that g is linear on the interval $[a,b]$, and since a and b are arbitrary real numbers with $a < b$, it follows that g is linear on all of $\mathbb{R}$. This completes the proof.