2019 UI MOCK PUTNAM CONTEST
October 12, 2019, 1 pm – 4 pm

1. Prove that, for any real numbers $a < b$,

$$\int_a^b (x^2+1)e^{-x^2}\,dx \geq e^{-a^2} - e^{-b^2}.$$

2. Given two nonnegative integers a and b, say that a **dominates** b if each digit in the binary expansion of a is greater or equal to the corresponding digit in the binary expansion of b. For example, the number $5 = (101)_2$ dominates the numbers $4 = (100)_2$, $1 = (1)_2$, and $0 = (0)_2$, but it does not dominate the numbers $3 = (11)_2$ or $2 = (10)_2$.

   Find, with proof, a simple general formula for the number of pairs (a, b) of integers in the interval $[0, 2^n)$ such that a dominates b.

3. Prove that, for $0 < x < \pi/4$,

$$(\cos x)^{\cos^2 x} > (\sin x)^{\sin^2 x}.$$

4. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \ldots$. Evaluate

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!}.$$

5. Consider a sequence of coins $C_1, C_2, \ldots$ such that coin C_n comes up heads with probability $1/n$. Let p_n be the probability of getting an *even* number of heads if coin C_n is flipped n times. Determine, with proof, $\lim_{n\to\infty} p_n$.

6. Determine, with proof, the set of all pairs (α, β) of positive real numbers for which the series

$$\sum_{m,n\in\mathbb{N}} \frac{(mn)^\alpha}{(m+n)^\beta}$$

   converges.

7. Find, with proof, all integrable functions $g : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^1 g(f(x))\,dx = g\left(\int_0^1 f(x)\,dx\right)$$

   for every integrable function $f : [0, 1] \to \mathbb{R}$.