1. Prove that, for any real numbers $a < b$,
\[ \int_a^b (x^2 + 1)e^{-x^2} \, dx \geq e^{-a^2} - e^{-b^2}. \]

2. Given two nonnegative integers $a$ and $b$, say that $a$ dominates $b$ if each digit in the binary expansion of $a$ is greater or equal to the corresponding digit in the binary expansion of $b$. For example, the number $5 = (101)_2$ dominates the numbers $4 = (100)_2$, $1 = (1)_2$, and $0 = (0)_2$, but it does not dominate the numbers $3 = (11)_2$ or $2 = (10)_2$.

Find, with proof, a simple general formula for the number of pairs $(a, b)$ of integers in the interval $[0, 2^n)$ such that $a$ dominates $b$.

3. Prove that, for $0 < x < \pi/4$,
\[ (\cos x)^{\cos^2 x} > (\sin x)^{\sin^2 x}. \]

4. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \ldots$. Evaluate
\[ \sum_{n=0}^{\infty} \frac{f_n(1)}{n!}. \]

5. Consider a sequence of coins $C_1, C_2, \ldots$ such that coin $C_n$ comes up heads with probability $1/n$. Let $p_n$ be the probability of getting an even number of heads if coin $C_n$ is flipped $n$ times. Determine, with proof, $\lim_{n \to \infty} p_n$.

6. Determine, with proof, the set of all pairs $(\alpha, \beta)$ of positive real numbers for which the series
\[ \sum_{m,n \in \mathbb{N}} \frac{(mn)^\alpha}{(m+n)^\beta} \]
converges.

7. Find, with proof, all integrable functions $g : \mathbb{R} \to \mathbb{R}$ such that
\[ \int_0^1 g(f(x)) \, dx = g \left( \int_0^1 f(x) \, dx \right) \]
for every integrable function $f : [0, 1] \to \mathbb{R}$.