2018 UI MOCK PUTNAM CONTEST
October 13, 2018, 1 pm – 4 pm
Solutions

1. Consider a quadratic polynomial of the form $P(x) = x^2 + ax + b$. If the coefficients a and b are randomly and independently selected from the interval $[0, 1]$, what is the probability that $P(x)$ has two real zeros?

   Solution. We will show that the desired probability is $\boxed{1/12}$. For the proof, note that the roots of $P(x)$ are $(1/2)(1 \pm \sqrt{a^2 - 4b})$. Thus, $P(x)$ has two real roots real if and only if $a^2 - 4b > 0$, i.e., if and only if $0 \le b < a^2/4$. Choosing a and b randomly and independently from $[0, 1]$ is equivalent to choosing the point (a, b) randomly from the unit square $0 \le a \le 1, 0 \le b \le 1$. Hence the desired probability is given by the area between the parabola $b = a^2/4$ and the a-axis over the interval $0 \le a \le 1$: This area is given by the integral $\int_0^1 (a^2/4)da = 1/12$.

2. Find, with proof, the smallest possible value of the sum

$$\sum_{i=1}^{n} \frac{a_i}{A - a_i},$$

   where $a_1, \ldots, a_n$ are positive real numbers and $A = \sum_{i=1}^{n} a_i$.

   Solution. We claim that the smallest value is $\boxed{n/(n-1)}$.
   Let S denote the given sum. If $a_i = 1$ for all i, then $A = n$ and $S = n/(n - 1)$, showing that the value $n/(n - 1)$ is attained. Thus it remains to show that $n/(n - 1)$ is always a lower bound for S. Setting $b_i = a_i/A$, we have $0 < b_i < 1$, $\sum_{i=1}^{n}(1 - b_i) = n - 1$, and

$$S = \sum_{i=1}^{n} \frac{b_i}{1 - b_i} = \sum_{i=1}^{n} \frac{1}{1 - b_i} - n.$$

   Applying AGM twice we get

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{1 - b_i} \ge \left(\prod_{i=1}^{n} \frac{1}{1 - b_i}\right)^{1/n} = \frac{1}{\left(\prod_{i=1}^{n}(1 - b_i)\right)^{1/n}} \ge \frac{1}{(1/n)\sum_{i=1}^{n}(1 - b_i)} = \frac{n}{n - 1}$$

   and hence

$$S \ge \frac{n^2}{n - 1} - n = \frac{n}{n - 1},$$

   which is the desired bound.

3. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

   Solution. Write the nth factor as a_n/b_n, where $a_n = n^3 - 1$ and $b_n = n^3 + 1$, and let $P_N = \prod_{n=2}^{N} a_n/b_n$ be the Nth partial product. Note that a_n and b_n factor into $a_n = (n - 1)(n^2 + n + 1)$

and $b_n = (n+1)(n^2-n+1) = (n+1)((n-1)^2+(n-1)+1)$, so P_N becomes a telescoping product that can be evaluated as follows:

$$P_N = \frac{\prod_{n=2}^{N}(n-1)(n^2+n+1)}{\prod_{n=2}^{N}(n+1)((n-1)^2+(n-1)+1)} = \frac{1\cdot 2\cdot (N^2+N+1)}{(1^2+1+1)\cdot N(N+1)}.$$

Hence

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1} = \lim_{N\to\infty} P_N = \lim_{N\to\infty}\frac{1\cdot 2\cdot (N^2+N+1)}{(1^2+1+1)\cdot N(N+1)} = \frac{2}{3}.$$

4. Call a set of positive integers *separated* if it does not contain two consecutive integers. For example, the set $\{1,3,7,9,14\}$ is separated, while the set $\{1,3,8,9,14\}$ is not separated (since it contains the consecutive integers 8 and 9). Let $f(n,k)$ be the number of k-element subsets of $\{1,2,\ldots,n\}$ that are separated. Find, with proof, a simple formula for $f(n,k)$.

Solution. We will show that $(*)$ $\boxed{f(n,k) = \binom{n-k+1}{k}}$.

Note that $\binom{n-k+1}{k}$ counts the number of k-element subsets of $\{1,\ldots,n-k+1\}$. Thus, to prove the claim, it suffices to establish a one-to-one correspondence between the set of separated k-element subsets of $\{1,\ldots,n\}$, and the set of all k-element subsets of $\{1,\ldots,n-k+1\}$.

Given a separated k-element subset $a_1 < \cdots < a_k$ of $\{1,\ldots,n\}$ let $a_i' = a_i + 1 - i$, $i = 1,\ldots,k$. Then $a_1' = a_1$, $a_{i+1}' - a_i' = a_{i+1} - a_i - 1 \geq 1$ for $i = 1,\ldots,k-1$, and $a_k' = a_k - k + 1 \leq n - k + 1$. Thus $a_1' < \cdots < a_k' \leq n-k+1$, so $\{a_1',\ldots,a_k'\}$ form a k-element subset of $\{1,\ldots,n-k+1\}$. Conversely, given any k-element subset $a_1' < \cdots < a_k' \leq n-k+1$ of $\{1,\ldots,n\}$, setting $a_i = a_i' + i - 1$, $i = 1,\ldots,k$, yields a separated k-element subset of $\{1,\ldots,n\}$. Thus we have a one-to-one correspondence between separated k-element subsets of $\{1,\ldots,n\}$ and unrestricted k-element subsets of $\{1,\ldots,n-k+1\}$.

5. Let $F(x)$ be a real-valued function satisfying $F(x) > x$ for all $x \geq 0$. Find, with proof, all non-zero polynomials P satisfying $P(0) = 0$ and $F(P(x)) = P(F(x))$ for all real numbers x.

Solution. We will show that $P(x) = x$ is the only such polynomial.

Obviously, $P(x) = x$ satisfies $P(0) = 0$ and $F(P(x)) = P(F(x))$ for all real numbers x, so our main task is to prove that there are no other solutions.

Assume $P(x)$ is a polynnmial satisfying the given identity. Let $a_0 = F(0)$, and $a_{n+1} = F(a_n)$ for $n \geq 0$. Then $P(a_0) = P(F(0)) = F(P(0)) = F(0) = a_0$. Then $P(a_1) = P(F(a_0)) = F(P(a_0)) = F(a_0) = a_1$, and inductively $P(a_n) = a_n$ for all n. Since $F(x) > x$ for $x \geq 0$, the sequence $\{a_n\}$ is strictly increasing, and, in particular, contains infinitely many elements. Thus, the polynomial $P(x) - x$ has infinitely many zeros and therefore must be identically 0. Hence $P(x) = x$ is the only polynomial satisfying the given identity.

6. Evaluate, with proof, the sum

$$\sum_{k=0}^{n}\binom{n+k}{k}2^{n-k}.$$

Solution. Let $S(n)$ denote the given sum. Working out the first few cases we are led to conjecture that $(*)$ $\boxed{S(n) = 2^{2n}}$ for all positive integers n. We give two proofs of $(*)$:

Proof I (Probabilistic argument): Dividing by 2^{2n} and using the identity $\binom{n+k}{k} = \binom{n+k}{n}$, the formula $(*)$ is equivalent to

$$(**) \qquad 1 = \sum_{k=0}^{n}\binom{n+k}{n}2^{-n-k}$$

We prove (∗∗) by interpreting the terms $\binom{n+k}{n}2^{-n-k}$, $k = 0, \ldots, n$, as a discrete probability distribution.

Consider two teams, A and B, playing a series of games against each other until one of the teams has won $n+1$ games. Assume that each team is equally likely to win a given game and that the games are independent of each other.

Let p_k denote the probability that the series ends in exactly $n+k+1$ games. Then $0 \le k \le n$, and the numbers $p_0, \ldots, p_k$ form a probability distribution. We will show that the numbers p_k are exactly the terms appearing in the sum on the right of (∗∗). Indeed, the series ends in exactly $n+k+1$ games if and only if either (a) team A wins game $n+k+1$ and wins exactly n out of the first $n+k$ games, or (b) team B wins game $n+k+1$ and wins exactly n out of the first $n+k$ games. Each of the two cases has probability $\binom{n+k}{n}2^{-n-k}(1/2)$, so the total probability for case (a) or (b) is $p_k = \binom{n+k}{n}2^{-n-k}$, as claimed.

Proof II (Induction): Let $S(n)$ denote the given sum. We will show by induction

For the base case, note that $S(1) = \binom{1}{0}2^1 + \binom{2}{1}2^0 = 2^2$, so (∗) holds for $n = 1$.

For the induction step, it suffices to show that $S(n+1) = 4S(n)$.

Writing $k = n+1+h$ and using the identity $\binom{n+1+h}{n+1} = \binom{n+1+h}{h} = \binom{n+h}{h} + \binom{n+h}{h-1}$, we have

$$\begin{aligned} S(n+1) &= \sum_{h=0}^{n+1}\binom{n+1+h}{h}2^{n+1-h} = \sum_{h=0}^{n+1}\binom{n+h}{h}2^{n+1-h} + \sum_{h=1}^{n+1}\binom{n+h}{h-1}2^{n+1-h} \\ &= \sum_{h=0}^{n}\binom{n+h}{h}2^{n+1-h} + \binom{2n+1}{n+1} + \sum_{h=0}^{n}\binom{n+1+h}{h}2^{n+1-h-1} \\ &= 2S(n) + \binom{2n+1}{n+1} + \frac{1}{2}S(n+1) - \frac{1}{2}\binom{2n+1}{n+1}. \end{aligned}$$

and hence

$$S(n+1) = 4S(n) + \left(2\binom{2n+1}{n+1} - \binom{2n+2}{n+1}\right).$$

Since $\binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n} = 2\binom{2n+1}{n+1}$, the term in parentheses is zero, so we have $S(n+1) = 4S(n)$, completing the induction step.

7. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\csc\frac{1}{n} - 1\right)^x$$

converges.

Solution. We will show that the series converges if and only if $\boxed{x > 1/2}$.

Let a_n be the expression in parentheses. Then

$$a_n = \frac{1}{n\sin(1/n)} - 1 = \frac{1 - n\sin(1/n)}{n\sin(1/n)}.$$

Using the Taylor expansion $\sin x = x - x^3/3! + O(x^5)$, we get

$$a_n = \frac{\frac{1}{6n^2} + O\left(\frac{1}{n^4}\right)}{1 - \frac{1}{6n^2} + O(\frac{1}{n^4})} = \frac{1}{6n^2}\cdot\frac{1 + O(1/n^2)}{1 - 1/(6n^2) + O(1/n^4)}.$$

It follows that, as $n \to \infty$, $a_n \sim 1/(6n^2)$, and hence $a_n^x \sim 1/(6^x n^{2x})$. Therefore the series $\sum_{n=1}^{\infty} a_n^x$ converges if and only if the series $\sum_{n=1}^{\infty} 1/n^{2x}$ converges. The latter series is a p-series, which converges if and only if the exponent is $2x > 1$, i.e., $x > 1/2$, as claimed.