2018 UI MOCK PUTNAM CONTEST
October 13, 2018, 1 pm – 4 pm

1. Consider a quadratic polynomial of the form $P(x) = x^2 + ax + b$. If the coefficients a and b are randomly and independently selected from the interval $[0, 1]$, what is the probability that $P(x)$ has two real zeros?

2. Find, with proof, the smallest possible value of the sum
$$\sum_{i=1}^{n} \frac{a_i}{A - a_i},$$
where $a_1, \ldots, a_n$ are positive real numbers and $A = \sum_{i=1}^{n} a_i$.

3. Evaluate the infinite product
$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

4. Call a set of positive integers *separated* if it does not contain two consecutive integers. For example, the set $\{1, 3, 7, 9, 14\}$ is separated, while the set $\{1, 3, 8, 9, 14\}$ is not separated (since it contains the consecutive integers 8 and 9). Let $f(n, k)$ be the number of k-element subsets of $\{1, 2, \ldots, n\}$ that are separated. Find, with proof, a simple formula for $f(n, k)$.

5. Let $F(x)$ be a real-valued function satisfying $F(x) > x$ for all $x \geq 0$. Find, with proof, all non-zero polynomials P satisfying $P(0) = 0$ and $F(P(x)) = P(F(x))$ for all real numbers x.

6. Evaluate, with proof, the sum
$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{n-k}.$$

7. Determine, with proof, the set of real numbers x for which
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$
converges.