2017 UI MOCK PUTNAM CONTEST
September 16, 2017, 1 pm – 4 pm
Solutions

1. (Virginia Tech Math Contest, 1995) Evaluate the integral

$$\int_0^1 \int_0^1 \frac{1}{1+\max(x,y)^2} dx dy,$$

where

$$\max(x,y) = \begin{cases} x & \text{if } x \geq y, \\ y & \text{if } x < y. \end{cases}$$

Solution. We claim that the integral is equal to $\boxed{\ln 2}$. For the proof, split the range of integration into two parts, corresponding to $x \leq y$ and $x > y$, respectively, and note that $\max(x,y) = y$ in the first part, and $\max(x,y) = x$ in the second part. By symmetry, the given integral is twice the integral over the first part. Thus,

$$\begin{aligned} \int_0^1 \int_0^1 \frac{1}{1+\max(x,y)^2} dx dy &= 2 \int_0^1 \int_0^y \frac{1}{1+\max(x,y)^2} dx dy \\ &= 2 \int_0^1 \int_0^y \frac{1}{1+y^2} dx dy \\ &= 2 \int_0^1 \frac{y}{1+y^2} dy = \int_1^2 \frac{1}{u} du = \ln 2. \end{aligned}$$

2. Call a permutation $a_1, a_2, \ldots, a_n$ of the integers $1, 2, \ldots, n$ a **polynomial permutation** of $1, 2, \ldots, n$ if exists a polynomial $P(x)$ with integer coefficients such that $P(k) = a_k$ for $k = 1, 2, \ldots, n$. Obviously, the identity permutation $1, 2, \ldots, n$ is always a polynomial permutation, corresponding to the polynomial $P(x) = x$.

   (a) Find, with proof, a *non-trivial* polynomial permutation (i.e., one that is not the identity permutation) of $1, 2, \ldots, 2017$.

   (b) For general n determine, with proof, *all* polynomial permutations of $1, 2, \ldots, n$.

Solution.

(a) The polynomial $P(x) = 2018 - x$ maps $1, 2, \ldots, 2017$ to $2017, 2016, \ldots, 1$, respectively. Thus the "reversal permutation" $2017, 2016, \ldots, 1$ is a non-trivial polynomial permutation of $1, 2, \ldots, 2017$.

(b) We will show that the identity permutation $a_k = k$ and the "reversal" permutation $a_k = n+1-k$ are the only polynomial permutations of $1, 2, \ldots, n$.
One direction is easy: The polynomials $P(x) = x$ and $P(x) = n + 1 - x$ show that these permutations are polynomial.
For the converse direction, suppose $P(x)$ is a polynomial with integer coefficients such that $P(k) = a_k$ for $k = 1, \ldots, n$, where $a_1, \ldots, a_n$ is a permutation of $1, \ldots, n$. From the congruence properties of polynomials we get

(1) $$(n-1) \mid P(n) - P(1) = a_n - a_1,$$
(2) $$(n-2) \mid P(n) - P(2) = a_n - a_2,$$
(3) $$(n-2) \mid P(n-1) - P(1) = a_{n-1} - a_1,$$
(4) $$(n-3) \mid P(n-1) - P(2) = a_{n-1} - a_2.$$

(1) implies that $|a_n - a_1| = n-1$, which is only possible if either $a_1 = 1$ and $a_n = n$ (Case I) or $a_1 = n$ and $a_n = 1$ (Case II).
In Case I, (3) implies $(n-2)|(a_{n-1}-1)$ and hence $a_{n-1} = n-1$, and (4) then gives $(n-3)|(n-1) - a_2$ and hence $a_2 = 2$. Continuing in this manner, we see that $a_k = k$ for all $k = 1, 2, \dots, n$, so the permutation $a_1, \dots, a_n$ is the identity permutation.
In Case II, (3) implies $(n-2)|(a_{n-1} - n)$ and hence $a_{n-1} = 2$, (4) then forces $a_2 = n-1$, and continuing in this manner, we obtain $a_k = n+1-k$ for $k = 1, 2, \dots, n$, so the permutation $a_1, \dots, a_n$ is the reversal permutation.
This completes the proof.

3. (A2, Putnam 2001) You have coins $C_1, C_2, \dots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. Let P_n be the probability that the number of heads is odd if all n coins are tossed. Find, with proof, a simple general formula for P_n.

Solution. We will prove by induction that $(*)$ $\boxed{P_n = n/(2n+1)}$.
If there is only a single coin C_1, the probability of obtaining an odd number of heads is the same as the probability that this coin lands heads, so we have $P_1 = 1/(2+1) = 1/3$, which proves $(*)$ for $n = 1$.
Now let $n \geq 1$ and suppose $(*)$ holds for n. Consider $n+1$ coins $C_1, C_2, \dots, C_{n+1}$ of the given type, let A_{n+1} be the event that there are an odd number of heads among all $n+1$ coins $C_1, \dots, C_{n+1}$, and let A_n be the event that there are an odd number of heads among the first n coins, $C_1, \dots, C_n$. Then

$$\begin{aligned} P_{n+1} &= P(A_{n+1}) = P(A_n)P(C_{n+1} \text{ lands tails}) + (1 - P(A_n))P(C_{n+1} \text{ lands heads}) \\ &= \frac{n}{2n+1} \cdot \left(1 - \frac{1}{2(n+1)+1}\right) + \left(1 - \frac{n}{2n+1}\right) \cdot \frac{1}{2(n+1)+1} \\ &= \frac{n+1}{2(n+1)+1} \end{aligned}$$

This proves $(*)$ for $n+1$ and completes the induction.

4. Given a real number α with $0 < \alpha < 1$, let

$$I_\alpha = \int_0^\infty \frac{dx}{x^\alpha(1+x)}.$$

Determine, with proof, the value of α for which the integral I_α is minimal.

Solution. We claim that $\boxed{\alpha = 1/2}$ minimizes I_α.
For the proof, split the given integral into two parts, one over the range $[0, 1]$ and the other over the range $[1, \infty)$. Changing variables $u = 1/x$ in the first integral converts this into an integral over $[1, \infty)$:

$$\int_0^1 \frac{dx}{x^\alpha(1+x)} = \int_1^\infty u^\alpha \frac{du}{(1+1/u)u^2} = \int_1^\infty u^{\alpha-1} \frac{du}{u+1}.$$

Combining this with the original integral over the range $[1, \infty)$, we obtain

$$I_\alpha = \int_1^\infty (x^{-\alpha} + x^{\alpha-1}) \frac{dx}{x+1}.$$

By the AGM inequality, we have

$$x^{-\alpha} + x^{\alpha-1} \geq 2\sqrt{x^{-\alpha}x^{\alpha-1}} = 2x^{-1/2}.$$

Hence,

$$I_\alpha \geq \int_1^\infty (x^{-1/2} + x^{1/2-1}) \frac{dx}{x+1} = I_{1/2},$$

so I_α is minimized at $\alpha = 1/2$, as claimed.

5. Let a, b be real numbers, and define a function $f(x)$ by

$$f(x) = \begin{cases} a & \text{if } \lfloor x \rfloor \text{ is odd,} \\ b & \text{if } \lfloor x \rfloor \text{ is even,} \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Find, with proof, the exact value of the series

$$\sum_{n=1}^{\infty} \frac{f(2^n \pi)}{2^n}$$

and express the result as a simple function of a and b.

Solution. Consider the binary expansion of π,

$$\pi = 2 + 1 + \sum_{k=1}^{\infty} \frac{a_k}{2^k},$$

where $a_k \in \{0, 1\}$. Then

$$\begin{aligned} \lfloor 2^n \pi \rfloor &= \lfloor 3 \cdot 2^n + a_1 2^{n-1} + \cdots + a_{n-1} 2^1 + a_n + a_{n+1} 2^{-1} + \cdots \rfloor \\ &= 3 \cdot 2^n + a_1 2^{n-1} + \cdots + a_{n-1} 2^1 + a_n. \end{aligned}$$

Hence $\lfloor 2^n \pi \rfloor$ is even if $a_n = 0$ and odd if $a_n = 1$. It follows that

$$f(2^n \pi) = b + a_n(a - b).$$

and therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{f(2^n \pi)}{2^n} &= \sum_{n=1}^{\infty} \frac{b + a_n(a-b)}{2^n} \\ &= b \sum_{n=1}^{\infty} \frac{1}{2^n} + (a - b) \sum_{n=1}^{\infty} \frac{a_n}{2^n} \\ &= b + (a - b)(\pi - 3) = \boxed{a(\pi - 3) + b(4 - \pi)}. \end{aligned}$$

6. For $d = 1, \ldots, 9$, let A_d denote the set of positive integers whose decimal representation contains only digits that are $\leq d$. For example, the number 310113 is in A_3 (and also in A_d for any $d \geq 3$) since all of its digits are ≤ 3.

For each $d \in \{1, 2, \ldots, 9\}$, determine, with proof, the precise set of positive real numbers p for which the series

$$\sum_{n \in A_d} \frac{1}{n^p}$$

converges.

Solution. We will show that the sum $\sum_{n \in A_d} 1/n^p$ converges if and only if $\boxed{p > \log_{10}(d+1)}$.

To prove this, we break the range of summation into finite intervals $I_k = 10^{k-1} \leq n < 10^k$, $k = 1, 2, \ldots$, and let

$$S_{k,d} = \sum_{n \in I_k \cap A_d} \frac{1}{n^p}$$

denote the sum over I_k. Now, note that the integers $n \in I_k$ are exactly those with k digits. Then $n \in A_d$ if and only if the first digit of n is in $\{1, \ldots, d\}$, while the remaining $k - 1$ digits are in $\{0, 1, \ldots, d\}$. Thus,

$$|I_k \cap A_d| = d \cdot (d + 1)^k.$$

On the other hand, if $n \in I_k$, then $10^{k-1} \le n < 10^k$. Thus,

$$S_{k,d} = \sum_{n \in I_k \cap A_d} \frac{1}{n^p} \begin{cases} \le \dfrac{d(d+1)^k}{10^{p(k-1)}}, \\ \ge \dfrac{d(d+1)^k}{10^{pk}}. \end{cases}$$

Hence

$$\sum_{n \in A_d} \frac{1}{n^p} = \sum_{k=1}^{\infty} S_{k,d} \le d \cdot 10^p \sum_{k=1}^{\infty} \left(\frac{d+1}{10^p} \right)^k.$$

If $p > \log_{10}(d+1)$, then $10^p > d+1$, so the latter series is a geometric series with ratio < 1 and hence converges. Thus, the series $\sum_{n \in A_d} 1/n^p$ converges when $p > \log_{10}(d+1)$.
Now suppose $p \le \log_{10}(d+1)$. Then $(d+1)/10^p \ge 1$, so $S_{k,d} \ge d$ and hence

$$\sum_{n \in A_d} \frac{1}{n^p} = \sum_{k=1}^{\infty} S_{k,d} \ge \sum_{k=1}^{\infty} d = \infty.$$

Thus, the series $\sum_{n \in A_d} 1/n^p$ diverges when $p \le \log_{10}(d+1)$. This completes the proof.