1. (Virginia Tech Math Contest, 1995) Evaluate the integral
\[ \int_0^1 \int_0^1 \frac{1}{1 + \max(x, y)^2} \, dx \, dy, \]
where
\[ \max(x, y) = \begin{cases} x & \text{if } x \geq y, \\ y & \text{if } x < y. \end{cases} \]

2. Call a permutation \( a_1, a_2, \ldots, a_n \) of the integers 1, 2, \ldots, \( n \) a polynomial permutation of 1, 2, \ldots, \( n \) if there exists a polynomial \( P(x) \) with integer coefficients such that \( P(k) = a_k \) for \( k = 1, 2, \ldots, n \). Obviously, the identity permutation 1, 2, \ldots, \( n \) is always a polynomial permutation, corresponding to the polynomial \( P(x) = x \).

(a) Find, with proof, a non-trivial polynomial permutation (i.e., one that is not the identity permutation) of 1, 2, \ldots, 2017.

(b) For general \( n \) determine, with proof, all polynomial permutations of 1, 2, \ldots, \( n \).

3. (A2, Putnam 2001) You have coins \( C_1, C_2, \ldots, C_n \). For each \( k \), \( C_k \) is biased so that, when tossed, it has probability \( 1/(2k+1) \) of falling heads. Let \( P_n \) be the probability that the number of heads is odd if all \( n \) coins are tossed. Find, with proof, a simple general formula for \( P_n \).

4. Given a real number \( \alpha \) with \( 0 < \alpha < 1 \), let
\[ I_\alpha = \int_0^\infty \frac{dx}{x^\alpha (1 + x)}. \]
Determine, with proof, the value of \( \alpha \) for which the integral \( I_\alpha \) is minimal.

5. Let \( a, b \) be real numbers, and define a function \( f(x) \) by
\[ f(x) = \begin{cases} a & \text{if } |x| \text{ is odd}, \\ b & \text{if } |x| \text{ is even}, \end{cases} \]
where \( |x| \) denotes the greatest integer \( \leq x \). Find, with proof, the exact value of the series
\[ \sum_{n=1}^\infty \frac{f(2^n \pi)}{2^n} \]
and express the result as a simple function of \( a \) and \( b \).

6. For \( d = 1, \ldots, 9 \), let \( A_d \) denote the set of positive integers whose decimal representation contains only digits that are \( \leq d \). For example, the number 310113 is in \( A_3 \) (and also in \( A_d \) for any \( d \geq 3 \)) since all of its digits are \( \leq 3 \).

For each \( d \in \{1, 2, \ldots, 9\} \), determine, with proof, the precise set of positive real numbers \( p \) for which the series
\[ \sum_{n \in A_d} \frac{1}{n^p} \]
converges.