2017 UI MOCK PUTNAM CONTEST
September 16, 2017, 1 pm – 4 pm

1. (Virginia Tech Math Contest, 1995) Evaluate the integral

$$\int_0^1 \int_0^1 \frac{1}{1+\max(x,y)^2} dxdy,$$

where

$$\max(x,y) = \begin{cases} x & \text{if } x \geq y, \\ y & \text{if } x < y. \end{cases}$$

2. Call a permutation $a_1, a_2, \ldots, a_n$ of the integers $1, 2, \ldots, n$ a **polynomial permutation** of $1, 2, \ldots, n$ if exists a polynomial $P(x)$ with integer coefficients such that $P(k) = a_k$ for $k = 1, 2, \ldots, n$. Obviously, the identity permutation $1, 2, \ldots, n$ is always a polynomial permutation, corresponding to the polynomial $P(x) = x$.

   (a) Find, with proof, a *non-trivial* polynomial permutation (i.e., one that is not the identity permutation) of $1, 2, \ldots, 2017$.

   (b) For general n determine, with proof, *all* polynomial permutations of $1, 2, \ldots, n$.

3. (A2, Putnam 2001) You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. Let P_n be the probability that the number of heads is odd if all n coins are tossed. Find, with proof, a simple general formula for P_n.

4. Given a real number α with $0 < \alpha < 1$, let

$$I_\alpha = \int_0^\infty \frac{dx}{x^\alpha(1+x)}.$$

Determine, with proof, the value of α for which the integral I_α is minimal.

5. Let a, b be real numbers, and define a function $f(x)$ by

$$f(x) = \begin{cases} a & \text{if } \lfloor x \rfloor \text{ is odd,} \\ b & \text{if } \lfloor x \rfloor \text{ is even,} \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Find, with proof, the exact value of the series

$$\sum_{n=1}^\infty \frac{f(2^n\pi)}{2^n}$$

and express the result as a simple function of a and b.

6. For $d = 1, \ldots, 9$, let A_d denote the set of positive integers whose decimal representation contains only digits that are $\leq d$. For example, the number 310113 is in A_3 (and also in A_d for any $d \geq 3$) since all of its digits are ≤ 3.

   For each $d \in \{1, 2, \ldots, 9\}$, determine, with proof, the precise set of positive real numbers p for which the series

$$\sum_{n \in A_d} \frac{1}{n^p}$$

converges.