2016 UI MOCK PUTNAM CONTEST
September 24, 2016, 1 pm – 4 pm
Solutions

1. Let F_n denote the nth Fibonacci number, defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n = 2, 3, \ldots$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}}$$

converges, and find its sum.

Solution. Since $F_{n+2} - F_n = F_{n+1}$, we have

$$\frac{1}{F_n F_{n+2}} = \frac{1}{F_{n+2} - F_n}\left(\frac{1}{F_n} - \frac{1}{F_{n+2}}\right) = \frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1} F_{n+2}}.$$

Thus, the series "telescopes", and we have for any positive integer N,

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+2}} = \frac{1}{F_1 F_2} - \frac{1}{F_{N+1} F_{N+2}}.$$

Since $F_N \to \infty$ as $N \to \infty$, the last terms goes to 0, so the given series converges with sum $1/(F_1 F_2) = 1$.

2. Let S be a finite set of positive integers, and let

$$f(x) = \sum_{n \in S} \frac{\sin(nx)}{n}.$$

Suppose that $|f(x)| \leq 2016|x|$ for all x. Prove that S can have at most 2016 elements.

Solution. We have, for any non-zero real number x,

$$\left| \sum_{n \in S} \frac{\sin(nx)}{nx} \right| = \left| \frac{f(x)}{x} \right| \leq 2016. \tag{1}$$

Now let $x \to 0$. Since

$$\lim_{x \to 0} \frac{\sin(nx)}{nx} = \lim_{x \to 0} \frac{n \cos(nx)}{n} = 1,$$

the left side in (1) converges to $|\sum_{n \in S} 1| = |S|$, while the right size remains equal to 2016. Hence, $|S| \leq 2016 + \epsilon$ for any $\epsilon > 0$, and since $|S|$ can only take integer values, we must have $|S| \leq 2016$, as claimed.

3. Let $\{x_n\}$, $n = 0, 1, 2, 3, \ldots$, be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1} x_{n+1} = 1 \quad \text{for } n = 1, 2, 3, \ldots$$

Prove that there exists a real number a such that

$$x_{n+1} = a x_n - x_{n-1} \quad \text{for } n = 1, 2, 3, \ldots.$$

Solution. Let $q_n = (x_{n+1} + x_{n-1})/x_n$. (Since the x_n are non-zero, q_n is well-defined.) Then $x_{n+1} = q_n x_n - x_{n-1}$, so to prove the claim, we need to show that q_n is constant. Now, from the given recurrence, we have, for all $n \geq 1$,

$$\begin{aligned} x_{n+1}^2 - x_n^2 &= x_n x_{n+2} - x_{n-1} x_{n+1}, \\ x_{n+1}(x_{n+1} + x_{n-1}) &= x_n(x_{n+2} + x_n), \\ \frac{x_{n+1} + x_{n-1}}{x_n} &= \frac{x_{n+2} + x_n}{x_{n+1}}, \end{aligned}$$

and hence $q_n = q_{n+1}$ for all $n \geq 1$. Thus, $q_n = q_1 = (x_2 + x_0)/x_1$, for all n, proving our claim.

Remark. This is Problem A2 in the 1993 Putnam. The Putnam book by Kedlaya, Poonen, and Vakil gives four different solutions.

4. Let $P(x) = \sum_{i=0}^{n} c_i x^i$ be a polynomial of degree $n \geq 2$ satisfying

$$c_n > 0,\ c_{n-1} > 0,\ c_{n-2} < 0,\ c_{n-3} < 0, \ldots,\ c_0 < 0.$$

Prove that $P(x)$ has at most one positive root.

Solution. Write $P(x)$ as $Q(x) - R(x)$, where $Q(x) = c_n x^n + c_{n-1}x^{n-1}$ and $R(x) = \sum_{i=0}^{n-2}(-c_i)x^i$. By assumption, the coefficients of $Q(x)$ and $R(x)$ are all positive.
Suppose $P(x)$ has two positive roots, $0 < x_1 < x_2$. Then $Q(x_1) = R(x_1)$ and $Q(x_2) = R(x_2)$.
Also, since $R(x)$ and $Q(x)$ have only positive coefficients, we have

$$\begin{aligned} Q(x_2) = c_n x_2^n + c_{n-1}x_2^{n-1} &\geq \left(\frac{x_2}{x_1}\right)^{n-1} \left(c_n x_1^n + c_{n-1} x_2^{n-1}\right) \\ &= \left(\frac{x_2}{x_1}\right)^{n-1} Q(x_1) = \left(\frac{x_2}{x_1}\right)^{n-1} R(x_1) \\ R(x_2) = \sum_{i=0}^{n-2} (-c_i) x_2^i &\leq \left(\frac{x_2}{x_1}\right)^{n-2} R(x_1). \end{aligned}$$

Hence $Q(x_2) \geq (x_2/x_1)R(x_2) > R(x_2)$, contradicting the relation $Q(x_2) = R(x_2)$. Thus, $P(x)$ can have at most one positive root.

Remark, This problem is related to *Decartes' Rule of Signs*, which relates the number of real zeros of a polynomial to the number of sign changes among its coefficients. See the Wikipedia article on this topic.

5. Suppose f is a non-negative continuous function on $[0, 1]$. Fix $a \in (0, 1)$. Prove that there exists $b \in [0, 1 - a]$ so that

$$\int_b^{b+a} f(t)\,dt \leq \frac{a}{1-a} \int_0^1 f(t)\,dt.$$

Solution. Let $I(b) = \int_b^{b+a} f(t)dt$ denote the integral on the left-hand side, and $I = \int_0^1 f(t)dt$ the integral on the right-hand side. Suppose that $I(b) > aI/(1 - a)$ for all $b \in [0, 1 - a]$. Then,

$$\begin{aligned} aI &< \int_0^{1-a} I(b)db = \int_0^{1-a} \int_b^{b+a} f(t)dt \\ &= \int_0^1 f(t) \int_{\max(0,t-a)}^{t} 1\,db\,dt \leq \int_0^1 f(t)a\,dt = aI, \end{aligned}$$

and we have obtained a contradiction. Thus, there must exist a $b \in [0, 1 - a]$ such that $I(b) \leq aI/(1 - a)$, as claimed.

6. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that

$$\sum_{i=1}^{n} a_i \le n^2 \quad \text{for } n = 1, 2, \ldots$$

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

Solution. By Cauchy's inequality and the given bound on $\sum_{i=1}^{n} a_i$, we have, for any integer $N \ge 1$,

$$\begin{aligned} N = \sum_{n=N+1}^{2N} \frac{1}{\sqrt{a_n}} \cdot \sqrt{a_n} &\le \left(\sum_{n=N+1}^{2N} \frac{1}{a_n} \right)^{1/2} \left(\sum_{n=N+1}^{2N} a_n \right)^{1/2} \\ &\le \left(\sum_{n=N+1}^{2N} \frac{1}{a_n} \right)^{1/2} \left(\sum_{n=1}^{2N} a_n \right)^{1/2} \le \left(\sum_{n=N+1}^{2N} \frac{1}{a_n} \right)^{1/2} \cdot (2N) \end{aligned}$$

and therefore

$$\sum_{n=N+1}^{2N} \frac{1}{a_n} \ge \left(\frac{N}{2N} \right)^2 = \frac{1}{4}.$$

Hence, the series $\sum_{n=1}^{\infty} 1/a_n$ does not satisfy Cauchy's criterion, and therefore diverges.

7. Let $A_1, A_2, \ldots, A_{2016}$ be sets satisfying:

   (i) Each set A_i contains exactly 20 elements.

   (ii) Each intersection $A_i \cap A_j$, $i \neq j$, of two distinct sets contains exactly one element.

Prove that there exists an element belonging to all 2016 sets.

Solution. The union $\bigcup_{i=1}^{2016} A_i$ has at most $20 \cdot 2016$ elements. For each such element e let $N(e)$ be the number of sets A_i that e belongs to, let e_0 be an element for which $N(e_0)$ is maximal, and let $N = N(e_0)$.

We seek to show that e_0 belongs to all 2016 sets, i.e., that $N = 2016$. Suppose that $N < 2016$. We will show that this leads to a contradiction.

Suppose first that $N \le 20$, so that $N(e) \le 20$ for all e. Pick a set A_1 from our collection. It shares each of the 20 members of A_1 with no more than 19 other sets, so the total number of sets cannot exceed $1 + 19 \cdot 20 < 2016$, a contradiction. Thus, we must have $N > 20$.

Now suppose $20 < N < 2016$. Denote the sets to which e_0 belongs by $A_1, \ldots, A_N$, and the sets not containing e_0 by $A_{N+1}, \ldots, A_{2016}$. For $1 \le i \le N$, denote by e_i the unique element in the intersection $A_i \cap A_{N+1}$. The elements $e_1, d_2, \ldots e_N$ are pairwise distinct, since if $e_i = e_j$ for some $i \neq j$, then $A_i \cap A_j$ would contain at least two elements, namely e_0 and e_i. It follows that A_{N+1} contains at least N distinct elements, namely $e_1, e_2, \ldots, e_N$. Hence $|A_{N+1}| \ge N > 20$, contradicting the assumption $|A_{N+1}| = 20$.

Hence N must be equal to 2016, and the proof is complete.