2016 UI MOCK PUTNAM CONTEST
September 24, 2016, 1 pm – 4 pm

1. Let F_n denote the nth Fibonacci number, defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n = 2, 3, \ldots$. Show that the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}}$$
converges, and find its sum.

2. Let S be a finite set of positive integers, and let
$$f(x) = \sum_{n \in S} \frac{\sin(nx)}{n}.$$
Suppose that $|f(x)| \le 2016|x|$ for all x. Prove that S can have at most 2016 elements.

3. Let $\{x_n\}$, $n = 0, 1, 2, 3, \ldots$, be a sequence of nonzero real numbers such that
$$x_n^2 - x_{n-1}x_{n+1} = 1 \quad \text{for } n = 1, 2, 3, \ldots$$
Prove that there exists a real number a such that
$$x_{n+1} = ax_n - x_{n-1} \quad \text{for } n = 1, 2, 3, \ldots.$$

4. Let $P(x) = \sum_{i=0}^{n} c_i x^i$ be a polynomial of degree $n \ge 2$ satisfying
$$c_n > 0,\ c_{n-1} > 0,\ c_{n-2} < 0,\ c_{n-3} < 0, \ldots,\ c_0 < 0.$$
Prove that $P(x)$ has at most one positive root.

5. Suppose f is a non-negative continuous function on $[0, 1]$. Fix $a \in (0, 1)$. Prove that there exists $b \in [0, 1-a]$ so that
$$\int_b^{b+a} f(t)\,dt \le \frac{a}{1-a} \int_0^1 f(t)\,dt.$$

6. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that
$$\sum_{i=1}^{n} a_i \le n^2 \quad \text{for } n = 1, 2, \ldots$$
Prove that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

7. Let $A_1, A_2, \ldots, A_{2016}$ be sets satisfying:
   (i) Each set A_i contains exactly 20 elements.
   (ii) Each intersection $A_i \cap A_j$, $i \ne j$, of two distinct sets contains exactly one element.

   Prove that there exists an element belonging to all 2016 sets.