2015 UI MOCK PUTNAM CONTEST
September 26, 2015, 1 pm – 4 pm
Solutions

1. Prove that a positive integer whose decimal representation contains each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly 3 times and does *not* contain the digit 8 (but with no restrictions on the number of the digits 0 and 9) cannot be a perfect square (i.e., a square of an integer).

   Solution. We use congruences modulo 9. Since an integer is congruent to its sum of digits modulo 9, the given number must be congruent to $3(1 + 2 + \cdots + 7) = 84 \equiv 3 \bmod 9$. On the other hand, the possible congruences of a perfect square modulo 9 are $0^2 = 0$, $(\pm 1)^2 \equiv 1$, $(\pm 2)^2 \equiv 4$, $(\pm 3)^2 \equiv 0$, and $(\pm 4)^2 \equiv 7$ modulo 9. Since $3 \bmod 9$ is is not on this list, the given number cannot be a perfect square.

2. Let $f(n)$ be the number of ordered pairs (x, y) of positive integers satisfying $n(x + y) = xy$.

   (a) Prove that, for any positive integer n, $f(n)$ is odd.

   (b) Find, with proof, a general formula for $f(2^n)$.

   Solution. (a) Note that the solutions (x, y) with $x \neq y$ come in pairs since (x, y) is a solution if and only if (y, x) is a solution. Thus, there are an even number of such solutions. On the other hand, there is exactly one solution with $x = y$, namely $x = 2n$. Therefore the total number of solutions, $f(n)$, is odd.
   (b) Rewrite the equation $n(x + y) = xy$ as $(*)$ $(x - n)(y - n) = n^2$. The two factors $x - n$ and $y - n$ must be either both positive or both negative. In the latter case, we have $1 \leq x < n$ and $1 \leq y < n$, so $(x-n)(y-n) = (n-x)(n-y) < n^2$, which contradicts $(*)$. Thus we necessarily have $x - n \geq 1$ and $y - n \geq 1$. Setting $a = x - n$ and $b = y - n$, the number, $f(n)$, of solutions to $(*)$ is seen to be equal to the number representations $n^2 = ab$ with $1 \leq a, b \leq n^2$. But the latter is equal to the number of divisors of n^2. In particular, since 2^{2n} has exactly $2n + 1$ positive divisors (namely, $2^0, 2^1, \ldots, 2^{2n}$), we have $f(2^n) = 2n + 1$.

3. Given a set $S = \{a_1, a_2, \ldots, a_k\}$ with $a_1 > a_2 > \ldots a_k$, define its *alternating sum* by $A(S) = a_1 - a_2 + a_3 - \cdots + (-1)^{k+1} a_k$. For example, $A(\{4\}) = 4$, $A(\{7, 3, 1\}) = 7 - 3 + 1 = 5$.

   Find, with proof, a simple general formula for the sum

$$\sum_{S \subset \{1,2,\ldots,n\}} A(S),$$

   where the summation is over all non-empty subsets of $\{1, 2, \ldots, n\}$.

   Solution. Let $f(n)$ denote the above sum. For convenience we define $A(\emptyset) = 0$ and include the empty set in the above summation. We will show that $(*)$ $f(n) = n2^{n-1}$.
   For $n = 1$, the sum reduces to $A(\{1\}) = 1$, which proves $(*)$ in this case. Now assume $n \geq 2$. The subsets $S \subset \{1, 2, \ldots, n\}$ are either of the form $S = S'$, where $S' \subset \{1, 2, \ldots, n - 1\}$, or of the form $S' \cup \{n\}$, where $S' \subset \{1, 2, \ldots, n - 1\}$. In the latter case, we have $A(S) = A(S' \cup \{n\}) = n - A(S')$. Hence,

$$\begin{aligned} f(n) &= \sum_{S' \subset \{1,2,\ldots,n-1\}} A(S') + \sum_{S' \subset \{1,2,\ldots,n-1\}} (n - A(S')) \\ &= \sum_{S' \subset \{1,2,\ldots,n-1\}} n = n2^{n-1}, \end{aligned}$$

   since there are 2^{n-1} subsets S' of $\{1, 2, \ldots, n - 1\}$. This proves $(*)$ for $n \geq 2$.

4. For any positive integer n, let $d(n)$ denote the *first* digit in the decimal representation of n. For example, $d(1) = 1$, $d(16) = 1$, $d(2015) = 2$. Determine, with proof, the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N : d(n^2) = 1\},$$

or show that the limit does not exist.

Solution. Let $A(N) = \#\{1 \le n \le N : d(n^2) = 1\}$. We claim that the limit $\lim_{N\to\infty} A(N)/N$ does **not** exist.

We argue by contradiction. Assume that the limit $\lim_{N\to\infty} A(N)/N$ exists and is equal to some nonnegative real number L. Let

$$N_k = \sqrt{10^k}, \quad M_k = \sqrt{2 \cdot 10^k},$$

so that $N_k^2 = 10^k$, $M_k^2 = 2 \cdot 10^k$, $N_{k+1}^2 = 10^{k+1}$. Then n^2 has leading digit 1 if and only if n falls into an interval of the type $[N_k, M_k)$, and it has a leading digit in $\{2, 3, \dots, 9\}$ if and only if n falls into an interval of the type $[M_k, N_{k+1})$. It follows that

$$A(M_k) \ge M_k - N_k - 1, \quad A(N_{k+1}) \le A(M_k) + 1. \tag{1}$$

The first inequality in (1) implies

$$L = \lim_{k\to\infty} \frac{A(M_k)}{M_k} \ge \lim_{k\to\infty} \frac{M_k - N_k - 1}{M_k} = 1 - \frac{1}{\sqrt{2}} \tag{2}$$

and hence $L > 0$. The second inequality in (1) implies

$$L = \lim_{k\to\infty} \frac{A(N_{k+1})}{N_{k+1}} \le \lim_{k\to\infty} \frac{A(M_k) + 1}{M_k} \lim_{k\to\infty} \frac{M_k}{N_{k+1}} = L \frac{\sqrt{2}}{\sqrt{10}}.$$

But this is only possible if $L = 0$, which contradicts (2). Hence, the limit $\lim_{N\to\infty} A(N)/N$ does not exist.

5. Let

$$P(n) = \prod_{k=1}^{n-1} (\sin(k\pi/n))^k.$$

Find, with proof, a simple formula for $P(n)$.

Solution. We will show that (1) $P(n) = 2^{-n(n-1)/2} n^{n/2}$. First, substituting $h = n - k$ gives

$$P(n) = \prod_{k=1}^{n-1} (\sin(k\pi/n))^k = \prod_{h=1}^{n-1} (\sin(n-h)\pi/n))^{n-h} = \prod_{h=1}^{n-1} (\sin(h\pi/n))^{n-h}.$$

Multiplying the two representations for $P(n)$, we get

$$P(n)^2 = \left(\prod_{k=1}^{n-1} \sin(k\pi/n) \right)^n = Q(n)^n,$$

say. To prove (1), it remains to show that (2) $Q(n) = n2^{-n+1}$.

Now,

$$\begin{aligned} Q(n) = \prod_{k=1}^{n-1} \sin(k\pi/n) &= \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \left| e^{ik\pi/n} - e^{-ik\pi/n} \right| \\ &= \frac{1}{2^{n-1}} \left| \prod_{k=1}^{n-1} \left(1 - e^{2\pi ik/n} \right) \right|. \end{aligned}$$

Now note that the numbers $z_k = e^{2\pi ik/n}$, $k = 1, 2, \ldots, n-1$, are the roots of the polynomial $(z^n - 1)/(z-1)$, so by the factorization theorem we have

$$\sum_{h=0}^{n-1} z^h = \frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1}(z - z_k).$$

Setting $z = 1$ we get $\prod_{k=1}^{n-1}(1 - z_k) = n$, and substituting this above we get (2).

6. Let a_n denote the number obtained by interpreting the decimal digits of n in base 11. (For example, $a_{137} = (137)_{11} = 1 \cdot 11^2 + 3 \cdot 11^1 + 7 = 161$.) Determine, with proof, all positive real values of p for which the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^p}$$

converges.

Solution. We will show that the series converges if and only if $p > \log 10/\log 11$.

For the proof, we break the summation range into intervals of the form

$$I_k = \{n : 11^{k-1} \le a_n < 11^k\},$$

and let $S_k = \sum_{a_n \in I_k} 1/a_n^p$ be the contribution of I_k to the given series.

Note that the integers $a_n \in I_k$ are exactly those of the form $a_n = \sum_{i=0}^{k-1} d_i 11^i$, with $d_{k-1} \in \{1, \ldots, 9\}$ and $d_i \in \{0, 1, \ldots, 9\}$ for $i = 0, 1, \ldots, k-2$. There are $9 \cdot 10^{k-1}$ such integers a_n, and for each such integer a_n we have $a_n^{-p} \le 11^{-(k-1)p}$ and $a_n^{-p} \ge 11^{-kp}$. Hence we have

$$\frac{9 \cdot 10^{k-1}}{11^{kp}} \le S_k \le \frac{9 \cdot 10^{k-1}}{11^{(k-1)p}},$$

Thus, the series $\sum_{k=1}^{\infty} S_k$ converges if and only if the series $\sum_{k=1}^{\infty} 10^k/11^{kp}$ converges. The latter series is a geometric series with ratio $10/11^p$, so it converges if and only if $10/11^p < 1$, i.e., if and only if $p > \log 10/\log 11$.

7. To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det\begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

Solution. (B3, Putnam 2015) We will show that $f(1) = 45$, $f(2) = 20250$, and $f(n) = 0$ for $n \ge 3$.

- For $n = 1$, the determinant sum is $\sum_{k=1}^{9} k = 45$.
- For $n = 2$, the sum is over all determinants $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ with $a = 1, 2, \ldots, 9$ and $b, c, d = 0, 1, 2, \ldots, 9$, i.e.,

$$\sum_{a=1}^{9}\sum_{b=0}^{9}\sum_{c=0}^{9}\sum_{d=0}^{9}(ac - bd) = 10 \cdot 10 \left(\sum_{a=1}^{9} a\right)\left(\sum_{d=0}^{9} d\right) - 9 \cdot 10 \left(\sum_{b=0}^{9} b\right)\left(\sum_{c=0}^{9} c\right)$$
$$= 100 \cdot 45^2 - 90 \cdot 45^2 = 20250.$$

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- For $n \ge 3$, the matrices in the sum in question can be matched up in pairs by interchanging the second and third rows. For each such pair of matrices the corresponding determinants cancel each other out, so the sum of all determinants is 0 when $n \ge 3$.