2015 UI MOCK PUTNAM CONTEST
September 26, 2015, 1 pm – 4 pm

1. Prove that a positive integer whose decimal representation contains each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly 3 times and does *not* contain the digit 8 (but with no restrictions on the number of the digits 0 and 9) cannot be a perfect square (i.e., a square of an integer).

2. Let $f(n)$ be the number of ordered pairs (x, y) of positive integers satisfying $n(x + y) = xy$.
   (a) Prove that, for any positive integer n, $f(n)$ is odd.
   (b) Find, with proof, a general formula for $f(2^n)$.

3. Given a set $S = \{a_1, a_2, \ldots, a_k\}$ with $a_1 > a_2 > \ldots a_k$, define its *alternating sum* by $A(S) = a_1 - a_2 + a_3 - \cdots + (-1)^{k+1}a_k$. For example, $A(\{4\}) = 4$, $A(\{7, 3, 1\}) = 7 - 3 + 1 = 5$.

   Find, with proof, a simple general formula for the sum
   $$\sum_{S\subset\{1,2,\ldots,n\}} A(S),$$
   where the summation is over all non-empty subsets of $\{1, 2, \ldots, n\}$.

4. For any positive integer n, let $d(n)$ denote the *first* digit in the decimal representation of n. For example, $d(1) = 1$, $d(16) = 1$, $d(2015) = 2$. Determine, with proof, the limit
   $$\lim_{N\to\infty} \frac{1}{N}\#\{1 \le n \le N : d(n^2) = 1\},$$
   or show that the limit does not exist.

5. Let
   $$P(n) = \prod_{k=1}^{n-1} (\sin(k\pi/n))^k.$$
   Find, with proof, a simple formula for $P(n)$.

6. Let a_n denote the number obtained by interpreting the decimal digits of n in base 11. (For example, $a_{137} = (137)_{11} = 1 \cdot 11^2 + 3 \cdot 11^1 + 7 = 161$.) Determine, with proof, all positive real values of p for which the series
   $$\sum_{n=1}^{\infty} \frac{1}{a_n^p}$$
   converges.

7. To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det\left(\begin{smallmatrix} 8 & 6 \\ 1 & 7 \end{smallmatrix}\right) = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

[Solutions at `http://www.math.uiuc.edu/contests.html`]