2014 UI MOCK PUTNAM CONTEST
September 27, 2014, 1 pm – 4 pm
Solutions

1. Let

$$f(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k} + \sqrt{k+1}}$$

Evaluate $f(9999)$.

Solution. Answer: $\boxed{99}$.
More generally, we will show that, for any positive integer n, $f(n) = \sqrt{n+1} - 1$. For the proof, rewrite the terms in the above sum as

$$\frac{1}{\sqrt{k} + \sqrt{k+1}} = \frac{\sqrt{k+1} - \sqrt{k}}{(k+1) - k} = \sqrt{k+1} - \sqrt{k},$$

to get a telescoping sum:

$$f(n) = \sum_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) = \sqrt{n+1} - 1.$$

2. Evaluate the integral

$$\int_0^\infty \frac{\ln x}{x^2 + 9} dx.$$

Solution. Answer: $\boxed{\pi(\ln 3)/6}$
More generally, we will show that, for any $a > 0$,

$$\int_0^\infty \frac{\ln x}{x^2 + a^2} dx = \frac{\pi \ln a}{2a}. \tag{1}$$

Let I denote the integral on the left of (1). Making the change of variables $x = ya$, we get

$$\begin{aligned} I &= \int_0^\infty \frac{\ln(ya)}{(ya)^2 + a^2} a\, dy = \frac{1}{a} \int_0^\infty \frac{\ln y + \ln a}{y^2 + 1} dy \\ &= \frac{1}{a} \int_0^\infty \frac{\ln y}{y^2 + 1} dy + \frac{\ln a}{a} \int_0^\infty \frac{1}{y^2 + 1} dy = \frac{1}{a} I_1 + \frac{\ln a}{a} I_2, \end{aligned}$$

say. The integral I_2 is easily evaluated: $I_2 = \arctan x\big|_0^\infty = \pi/2$, and multiplying with the coefficient $(\ln a)/a$ gives the expression on the right of (1). On the other hand, the following calculation shows that the integral I_2 is equal to 0:

$$\begin{aligned} I_2 &= \int_0^1 \frac{\ln y}{y^2 + 1} dy + \int_1^\infty \frac{\ln y}{y^2 + 1} dy \\ &= \int_0^1 \frac{\ln y}{y^2 + 1} dy + \int_0^1 \frac{\ln(1/z)}{(1/z)^2 + 1} \cdot \frac{1}{z^2} dz \quad \text{(set } z = 1/y\text{)} \\ &= \int_0^1 \frac{\ln y}{y^2 + 1} dy + \int_0^1 \frac{-\ln z}{1 + z^2} dz = 0. \end{aligned}$$

3. Show that a positive integer whose decimal representation contains each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly 3 times and does *not* contain the digit 8 (but with no restrictions on the number of the digits 0 and 9) cannot be a perfect square.

   Solution. We use congruences modulo 9. Since an integer is congruent to its sum of digits modulo 9, the given number must be congruent to $3(1+2+\cdots+7) = 84 \equiv 3 \bmod 9$. On the other hand, the possible congruences of a perfect square modulo 9 are $0^2 = 0$, $(\pm 1)^2 \equiv 1$, $(\pm 2)^2 \equiv 4$, $(\pm 3)^2 \equiv 0$, and $(\pm 4)^2 \equiv 7$ modulo 9. Since $3 \bmod 9$ is is not on this list, the given number cannot be a perfect square.

4. (Virginia Tech Math Contest 2007) Let n be a positive integer, let A, B be symmetric $n \times n$ matrices with real entries. Suppose there exist $n \times n$ matrices X, Y such that $\det(AX + BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$.

   Solution. We argue by contradiction. Thus, assume that $\det(A^2 + B^2) = 0$. Then there exists a nonzero n-dimensional (column) vector $\mathbf{u}$ that satisfies $(A^2 + B^2)\mathbf{u} = \mathbf{0}$, where $\mathbf{0}$ denotes the n-dimensional zero vector.
   Now multiply each side of this equation from the left by $\mathbf{u}^T$, the transpose of the vector $\mathbf{u}$ (which is a row vector of dimension n, i.e., a $1 \times n$ matrix). We get

$$\mathbf{u}^T(A^2 + B^2)\mathbf{u} = 0, \tag{1}$$

   where now the left and right sides are both scalar quantities. Since the matrix A is symmetric, we have $A^T = A$. Thus, using the properties of a transpose, we have

$$\mathbf{u}^T A^2 \mathbf{u} = \mathbf{u}^T A^T (A\mathbf{u}) = (A\mathbf{u})^T (A\mathbf{u}) = |A\mathbf{u}|^2,$$
$$\mathbf{u}^T B^2 \mathbf{u} = \mathbf{u}^T B^T (B\mathbf{u}) = (B\mathbf{u})^T (B\mathbf{u}) = |B\mathbf{u}|^2,$$

   where $|\ldots|$ denotes the usual norm (i.e., magnitude) of a vector. Substituting this into (1) gives

$$|A\mathbf{u}|^2 + |B\mathbf{u}|^2 = 0,$$

   which forces $|A\mathbf{u}| = |B\mathbf{u}| = 0$, and hence $A\mathbf{u} = \mathbf{0}$ and $B\mathbf{u} = \mathbf{0}$. But then $(X^T A + Y^T B)\mathbf{u} = \mathbf{0}$, so the matrix $X^T A + Y^T B$, and hence also its transpose, $A^T X + B^T Y = AX + BY$, must have determinant 0, contradicting the assumption $\det(AX + BY) \neq 0$.

5. (B3, Putnam 2001) For any real number t, let $\langle t \rangle$ denote the integer closest to t; for example, $\langle 3.14159 \rangle = 3$ and $\langle 2.71828 \rangle = 3$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle \sqrt{n} \rangle} + 2^{-\langle \sqrt{n} \rangle}}{2^n}.$$

   Solution. Answer: $\boxed{3}$
   For the proof, note first that $\langle \sqrt{n} \rangle$ takes on the values $1, 2, 3, \ldots$, and for each such value k we have $\langle \sqrt{n} \rangle = k$ if and only if $(k - 1/2) \leq \sqrt{n} \leq (k + 1/2)$, i.e., $k^2 - k + 1/4 \leq n \leq k^2 + k + 1/4$. Since n is an integer, the latter condition is equivalent to $k^2 - k + 1 \leq n \leq k^2 + k$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{\langle \sqrt{n} \rangle} + 2^{-\langle \sqrt{n} \rangle}}{2^n} &= \sum_{k=1}^{\infty} \sum_{n=k^2-k+1}^{k^2+k} \frac{2^k + 2^{-k}}{2^n} \\
&= \sum_{k=1}^{\infty} (2^k + 2^{-k})(2^{-k^2+k} - 2^{-k^2-k}) \\
&= \sum_{k=1}^{\infty} (2^{-k(k-2)} - 2^{-k(k+2)}) \\
&= \sum_{k=1}^{\infty} 2^{-k(k-2)} - \sum_{h=3}^{\infty} 2^{-(h-2)h} = 3.
\end{aligned}$$

6. Let $a_1, a_2, a_3, \ldots$ be positive real numbers, and let $S_n = \sum_{k=1}^{n} a_k$.

(a) (Virginia Tech Math Contest, 1979) Prove or disprove: If the series $\sum_{n=1}^{\infty} a_n$ *diverges*, then the series $\sum_{n=1}^{\infty} \frac{a_n}{S_n^2}$ *converges.*

Solution. We claim that the series $\sum_{n=1}^{\infty} a_n/S_n^2$ converges.
Proof: By the assumption that $a_n > 0$ all terms in this series are positive, so it suffices to show that its partial sums are bounded. Now, using $S_n - S_{n-1} = a_n > 0$, we have for $N \geq 2$,

$$\begin{aligned}
\sum_{n=1}^{N} \frac{a_n}{S_n^2} &= \frac{1}{a_1} + \sum_{n=2}^{N} \frac{S_n - S_{n-1}}{S_n^2} \\
&\leq \frac{1}{a_1} + \sum_{n=2}^{N} \left(\frac{S_n - S_{n-1}}{S_n S_{n-1}} \right) \\
&= \frac{1}{a_1} + \sum_{n=2}^{N} \frac{1}{S_{n-1}} - \frac{1}{S_n} \\
&= \frac{1}{a_1} + \frac{1}{S_1} - \frac{1}{S_N} \leq \frac{2}{a_1}.
\end{aligned}$$

Hence the partial sums of the series $\sum_{n=1}^{\infty} a_n/S_n^2$ are bounded by $2/a_1$, and the series therefore converges.

(b) (A5, Putnam 1964, variation) Prove or disprove: If the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, then the series $\sum_{n=1}^{\infty} \frac{n}{S_n}$ converges as well.

Solution. We claim that the series $\sum_{n=1}^{\infty} n/S_n$ converges.
Proof: For any nonnegative integer k, set

$$A_k = \sum_{2^k \leq i < 2^{k+1}} a_i, \quad B_k = \sum_{2^k \leq i < 2^{k+1}} \frac{1}{a_i}.$$

By the Cauchy-Schwarz inequality,

$$2^{2k} = \left(\sum_{2^k \leq i < 2^{k+1}} \sqrt{a_i} \cdot \frac{1}{\sqrt{a_i}} \right)^2 \leq A_k B_k,$$

so

$$\frac{2^{2k}}{A_k} \leq B_k.$$

On the other hand, for $2^{k+1} \leq n < 2^{k+2}$ we have

$$\frac{n}{S_n} = \frac{n}{a_1 + a_2 + \cdots + a_n} \leq \frac{n}{A_k} \leq \frac{2^{k+2}}{A_k}$$

and hence

$$\sum_{2^{k+1} \leq n < 2^{k+2}} \frac{n}{S_n} \leq 2^{k+1} \frac{2^{k+2}}{A_k} = \frac{2^{2k+3}}{A_k} \leq 2^3 B_k = 8 \sum_{2^k \leq i < 2^{k+1}} \frac{1}{a_i}.$$

Summing over all integers $k \geq 0$ gives

$$\sum_{2 \leq n < \infty} \frac{n}{S_n} \leq 8 \sum_{1 \leq i < \infty} \frac{1}{a_i}.$$

Thus, if $\sum_{i=1}^{\infty} 1/a_i$ converges, then so does the series $\sum_{n=1}^{\infty} n/S_n$