2014 UI MOCK PUTNAM CONTEST
September 27, 2014, 1 pm – 4 pm

1. Let

$$f(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k} + \sqrt{k+1}}$$

Evaluate $f(9999)$.

2. Evaluate the integral

$$\int_0^\infty \frac{\ln x}{x^2+9} dx.$$

3. Show that a positive integer whose decimal representation contains each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly 3 times and does *not* contain the digit 8 (but with no restrictions on the number of the digits 0 and 9) cannot be a perfect square.

4. (Virginia Tech Math Contest 2007) Let n be a positive integer, let A, B be symmetric $n \times n$ matrices with real entries. Suppose there exist $n \times n$ matrices X, Y such that $\det(AX+BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$.

5. (B3, Putnam 2001) For any real number t, let $\langle t \rangle$ denote the integer closest to t; for example, $\langle 3.14159 \rangle = 3$ and $\langle 2.71828 \rangle = 3$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle \sqrt{n} \rangle} + 2^{-\langle \sqrt{n} \rangle}}{2^n}.$$

6. Let $a_1, a_2, a_3, \ldots$ be positive real numbers, and let $S_n = \sum_{k=1}^{n} a_k$.

   (a) (Virginia Tech Math Contest, 1979) Prove or disprove: If the series $\sum_{n=1}^{\infty} a_n$ *diverges*, then the series $\sum_{n=1}^{\infty} \frac{a_n}{S_n^2}$ *converges*.

   (b) (A5, Putnam 1964, variation) Prove or disprove: If the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, then the series $\sum_{n=1}^{\infty} \frac{n}{S_n}$ converges as well.

[Solutions at `http://www.math.uiuc.edu/contests.html`]