2013 UI MOCK PUTNAM EXAM
September 25, 2013, 5 pm – 7 pm
Solutions

1. Let $f(n)$ denote the n-th term in the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \dots$, obtained by writing one 1, two 2's, three 3's, four 4's, etc.

   (a) Find, with proof, $f(2013)$.

   (b) Find, with proof, a simple general formula for $f(n)$. (The formula can involve the floor or ceiling function.)

   Solution. Answer: (a) $\boxed{f(2013) = 63}$; (b) $\boxed{f(n) = \lceil \sqrt{2n + 1/4} - 1/2 \rceil}$.

   (a) The first k blocks occupy $1 + 2 + \dots + k = k(k+1)/2$ positions. Thus, we have $f(n) = k$ if and only if n occurs among the last k of these positions, i.e., if and only if

   $$\frac{k(k-1)}{2} < n \le \frac{k(k+1)}{2}. \tag{1}$$

   For the case $n = 2013$, a direct calculation gives the appropriate k-value: $62 \cdot 63/2 = 63 \cdot 31 = 1953$ and $63 \cdot 64/2 = 63 \cdot 32 = 2016$, so the k-value satisfying (1) is $k = 63$, and hence $f(n) = 63$.

   (b) To get a general formula, we multiply (1) by 2 and complete the square on both sides:

   $$\left(k - \frac{1}{2}\right)^2 < 2n + \frac{1}{4} \le \left(k + \frac{1}{2}\right)^2,$$
   $$k - 1 < \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \le k.$$

   This shows that $f(n) = k = \lceil \sqrt{2n + 1/4} - 1/2 \rceil$.

2. (Problem B1, Putnam 1971) Let S be a set and $*$ a binary operation on S satisfying $x * x = x$ for all $x \in S$ and $(x * y) * z = (y * z) * x$ for all $x, y, z \in S$. Prove that $*$ is commutative, i.e., that $x * y = y * x$ holds for all $x, y \in S$.

   Solution. Applying the two given rules repeatedly, we have, for any $x, y \in S$,

   $$\begin{aligned} x * y &= (x * y) * (x * y) = ((x * y) * x) * y = ((y * x) * x) * y = ((x * x) * y) * y \\ &= (x * y) * y = (y * y) * x = y * x, \end{aligned}$$

   which proves the commutativity of $*$.

3. (Problem B1, Putnam 1980) For which real numbers c is

   $$\frac{1}{2}\left(e^x + e^{-x}\right) \le e^{cx^2}$$

   for all real numbers x? Prove your answer.

   Solution. Answer: $\boxed{c \ge 1/2}$

   To show that any $c \ge 1/2$ "works", expand both sides into a Taylor series and compare coefficients of corresponding powers of x:

$$(1) \qquad \frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!} = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!},$$

$$(2) \qquad e^{cx^2} = \sum_{m=0}^{\infty} \frac{c^m x^{2m}}{m!}.$$

The constant terms on both sides are equal, and for $m \geq 1$, the coefficients of x^{2m} in (1) and (2) are $1/(2m)!$ and $c^m/m!$, respectively. Since $(2m)! = m!(m+1)\dots(2m) \geq 2^m m!$, we have $1/(2m)! \leq c^m/m!$, for $m \geq 1$, when $c \geq 1/2$, so all terms in the series (1) are less than or equal to the corresponding terms in the series (2). This proves that the inequality holds when $c \geq 1/2$.

To show that $c < 1/2$ does not work, we consider the behavior of both sides when $x \to 0$: From (1) and (2) we see that

$$e^{cx^2} - \frac{1}{2}(e^x + e^{-x}) = \left(1 + \frac{cx^2}{1!} + O(x^4)\right) - \left(1 + \frac{x^2}{1!} + O(x^4)\right) = \frac{2c-1}{2}x^2 + O(x^4),$$

and hence

$$\lim_{x\to 0} \frac{1}{x^2}\left(e^{cx^2} - \frac{1}{2}(e^x + e^{-x})\right) = \frac{2c-1}{2} < 0,$$

when $c < 1/2$. Thus, if $c < 1/2$, the inequality fails when x is sufficiently close to 0.

Alternative solution (Krishna Srinivasan): First note that, since both sides of the inequality are even functions of x, we may restrict to values $x \geq 0$.

Taking logarithms on both sides, we can rewrite the inequality as $cx^2 \geq \ln\cosh(x)$, or

$$(1) \qquad f(x) = cx^2 - \ln\cosh(x) \geq 0.$$

Now compute

$$(2) \qquad f'(x) = 2cx - \frac{\sinh x}{\cosh x},$$

$$(3) \qquad f''(x) = 2c - \frac{1}{(\cosh x)^2}$$

At $x = 0$, $f(0) = 0 - \ln 1 = 0$ by (1), and $f'(0) = 0$ by (2).

If $c \geq 1/2$, then (3) gives $f''(x) \geq 1 - 1/(\cosh x)^2 \geq 1 - 1 \geq 0$ for $x \geq 0$, so $f'(x)$ is nondecreasing for $x \geq 0$, and hence $f'(x) \geq f'(0) = 0$. Therefore $f(x) \geq f(0) = 0$ for $x \geq 0$, so (1) holds for all $x \geq 0$ in the case $c \geq 1/2$.

Conversely, if $c < 1/2$, then (3) shows $f''(0) = 2c - 1 < 0$, so $f'(x)$ is decreasing for sufficiently small x, and hence $f'(x) < f'(0) = 0$ and $f(x) < 0$ for such x. Thus, (1) fails for small enough x in the case $c < 1/2$.

4. Find, with proof, a simple formula for the sum

$$\sum_{k=n}^{2n} \binom{k}{n} 2^{2n-k}.$$

Solution. Answer: $\boxed{4^n}$

Let $S(n)$ denote the given sum. We will show by induction that

$$(*) \qquad S(n) = 2^{2n}.$$

For $n=1$, we have $S(1)=\binom{1}{0}+\binom{2}{1}2^{-1}=2$, so $(*)$ holds in this case. Now let $n\ge 1$ and suppose $(*)$ holds for this n. Then, using the recurrence formula $\binom{k}{n}=\binom{k-1}{n}+\binom{k-1}{n-1}$ we get

$$\begin{aligned}
S(n+1) &= \sum_{k=n+1}^{2(n+1)} \binom{k}{n+1} 2^{2n+2-k} \\
&= \sum_{k=n+2}^{2n+2} \binom{k-1}{n+1} 2^{2n+2-k} + \sum_{k=n+1}^{2n+2} \binom{k-1}{n} 2^{2n+2-k} \\
&= \sum_{h=n+1}^{2n+1} \binom{h}{n+1} 2^{2n+1-h} + \sum_{h=n}^{2n+1} \binom{h}{n} 2^{2n+1-h} \quad \text{(set } h=k-1\text{)} \\
&= \frac{1}{2}\left(S(n+1) - \binom{2n+2}{n+1} \right) + 2\left(S(n) + \frac{1}{2}\binom{2n+1}{n} \right) \\
&= \frac{1}{2} S(n+1) + 2S(n),
\end{aligned}$$

since

$$\binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{2n+2}{n+1}\binom{2n+1}{n} = 2\binom{2n+1}{n}.$$

Hence $S(n+1)=4S(n)=4\cdot 4^n=4^{n+1}$. This proves the formula for $n+1$ and completes the induction.

5. (B3, Putnam 1982) Let $f(n)$ be the number of ordered pairs (a,b) of integers from $\{1,2,\ldots,n\}$ such that $a+b$ is a perfect square (i.e., of the form k^2, for some integer k). Prove that the limit $\lim_{n\to\infty} f(n)n^{-3/2}$ exists and express this limit in the form $r(\sqrt{s}-t)$, where s and t are integers and r is a rational number.

Solution. Answer: $\boxed{\frac{4}{3}(\sqrt{2}-1)}$.

By definition, $f(n)$ is the number of pairs (a,b) of integers from $\{1,2,\ldots,n\}$ such that $a+b=k^2$ for some positive integer k. We first count the number of such pair for fixed k.

- If $k^2\le n$, then there are exactly k^2-1 such pairs, namely $(1,k^2-1),(2,k^2-2),\ldots,(k^2-1,1)$.
- If $n<k^2\le 2n$, then the possible pairs are $(k^2-n,n),(k^2-n+1,n-1),\ldots,(n,k^2-n)$, so there are $n-(k^2-n)+1=2n+1-k^2$ such pairs in this case.
- If $k^2>2n$, there are no such pairs.

Adding up these counts, we get

$$\begin{aligned}
(1)\qquad f(n) &= \sum_{k=1}^{\lfloor\sqrt{n}\rfloor}(k^2-1) + \sum_{k=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\sqrt{2n}\rfloor}(2n+1-k^2) \\
&= S(\lfloor\sqrt{n}\rfloor) - \lfloor\sqrt{n}\rfloor + (2n+1)\left(\lfloor\sqrt{2n}\rfloor - \lfloor\sqrt{n}\rfloor\right) - S(\lfloor\sqrt{2n}\rfloor) + S(\lfloor\sqrt{n}\rfloor),
\end{aligned}$$

where $S(t)=\sum_{k=1}^{t}k^2$. Now,

$$\lim_{t\to\infty}\frac{S(t)}{t^3} = \lim_{t\to\infty}\frac{t(t+1)(2t+1)}{6t^3} = \frac{1}{3}.$$

Hence, dividing (1) by $n^{3/2}$ and letting $n\to\infty$, we get

$$\lim_{n\to\infty}\frac{f(n)}{n^{3/2}} = \frac{1}{3} - 0 + 2(\sqrt{2}-1) - \frac{\sqrt{2}^3}{3} + \frac{1}{3} = \frac{4}{3}(\sqrt{2}-1).$$

6. Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be sequences of positive real numbers satisfying

(1) $$y_1 \geq y_2 \geq y_3 \geq \cdots,$$

and

(2) $$x_1 x_2 \ldots x_k \geq y_1 y_2 \ldots y_k$$

for $k = 1, 2, 3 \ldots$. Prove that

$$x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k$$

for $k = 1, 2, 3 \ldots$.

Solution. From (2) and the AGM inequality, we get

(3) $$\sum_{i=1}^{k} \frac{x_i}{y_i} \geq k \left(\frac{x_1}{y_1} \ldots \frac{x_k}{y_k} \right)^{1/k} \geq k$$

Set $d_i = (x_i - y_i)/y_i$. From (3) we get

(4) $$S_k := \sum_{i=1}^{k} d_i = \sum_{i=1}^{k} \frac{x_i}{y_i} - k \geq 0.$$

On the other hand, we have (setting $S_0 = 0$)

$$\begin{aligned} \sum_{i=1}^{k} (x_i - y_i) &= \sum_{i=1}^{k} d_i y_i \\ &= \sum_{i=1}^{k} (S_i - S_{i-1}) y_i \\ &= \sum_{i=1}^{k-1} S_i (y_i - y_{i+1}) + S_k y_k \geq 0, \end{aligned}$$

since $S_i \geq 0$ by (4) and $y_i - y_{i+1} \geq 0$ by (1). Thus,

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \quad (k = 1, \ldots, n),$$

as claimed.