2013 UI MOCK PUTNAM EXAM
September 25, 2013, 5 pm – 7 pm

1. Let $f(n)$ denote the n-th term in the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \dots$, obtained by writing one 1, two 2's, three 3's, four 4's, etc.

   (a) Find, with proof, $f(2013)$.

   (b) Find, with proof, a simple general formula for $f(n)$. (The formula can involve the floor or ceiling function.)

2. (Problem B1, Putnam 1971) Let S be a set and $*$ a binary operation on S satisfying $x * x = x$ for all $x \in S$ and $(x * y) * z = (y * z) * x$ for all $x, y, z \in S$. Prove that $*$ is commutative, i.e., that $x * y = y * x$ holds for all $x, y \in S$.

3. (Problem B1, Putnam 1980) For which real numbers c is

$$\frac{1}{2}\left(e^x + e^{-x}\right) \le e^{cx^2}$$

   for all real numbers x? Prove your answer.

4. Find, with proof, a simple formula for the sum

$$\sum_{k=n}^{2n} \binom{k}{n} 2^{2n-k}.$$

5. (B3, Putnam 1982) Let $f(n)$ be the number of ordered pairs (a, b) of integers from $\{1, 2, \dots, n\}$ such that $a + b$ is a perfect square (i.e., of the form k^2, for some integer k). Prove that the limit $\lim_{n\to\infty} f(n)n^{-3/2}$ exists and express this limit in the form $r(\sqrt{s} - t)$, where s and t are integers and r is a rational number.

6. Let $x_1, x_2, x_3, \dots$ and $y_1, y_2, y_3, \dots$ be sequences of positive real numbers satisfying

$$y_1 \ge y_2 \ge y_3 \ge \cdots, \tag{1}$$

   and

$$x_1 x_2 \dots x_k \ge y_1 y_2 \dots y_k \tag{2}$$

   for $k = 1, 2, 3 \dots$. Prove that

$$x_1 + x_2 + \cdots + x_k \ge y_1 + y_2 + \cdots + y_k$$

   for $k = 1, 2, 3 \dots$.

[Solutions at `http://www.math.uiuc.edu/contests.html`]