2012 UI MOCK PUTNAM EXAM

Solutions

1. Prove that, given any power of 2 (such as 1024), there exist infinitely many powers of 2 whose decimal representation ends with the digits of the given power of 2.

   Solution. Since two integers end in the same last k digits if and only if they are congruent modulo 10^k, it suffices to show that, given any positive integer k, there exist infinitely many n such that $2^n \equiv 2^k \bmod 10^k$. For $n \geq k$ the latter congruence is equivalent to $(*)$ $2^{n-k} \equiv 1 \bmod 5^k$. By the Euler-Fermat theorem, we have $2^{\phi(5^k)} \equiv 1 \bmod 5^k$, where $\phi(5^k) = 4 \cdot 5^{k-1}$ is the Euler Phi function. Hence $(*)$ holds whenever n is of the form $n = k + 4 \cdot 5^{k-1}m$ for some nonnegative integer m.

2. Determine, with proof, all positive integers n for which there is a polynomial of degree n satisfying the following three conditions:

   (i) $P(k) = k$ for $k = 1, 2, \ldots, n$;

   (ii) $P(0)$ is an integer;

   (iii) $P(-1) = 2012$.

   Solution. We will show that the positive integers n for which a polynomial with the stated properties exists are exactly those of the form $n = d - 1$, where d is a divisor of $2012 + 1$. Now $2013 = 3 \cdot 11 \cdot 61$, so the values of d are $1, 3, 11, 33, 61, 183, 671, 2013$, with $\boxed{2, 10, 32, 60, 182, 670, 2012}$ as the corresponding values of n for which a polynomial of the desired form exists..

   Suppose first that $P(x)$ is a polynomial of degree n satisfying the three conditions (i), (ii), and (iii). Consider the polynomial $Q(x) = P(x) - x$. Then $Q(x)$ has degree at most n, and condition (i) implies that $Q(x)$ has a root at each of the numbers $k = 1, 2, \ldots, n$. It follows that $Q(x)$ is of the form $Q(x) = C(x-1)(x-2)\ldots(x-n)$ for some constant C. Now,

$$2012 = P(-1) = Q(-1) + (-1) = C(-1)^n(n+1)! - 1 \quad \text{(by (iii))}, \tag{1}$$

$$C = \frac{2013(-1)^n}{(n+1)!} \tag{2}$$

$$P(0) = Q(0) = C(-1)^n n! = \frac{2013}{n+1} \tag{3}$$

$$P(x) = C(x-1)(x-2)\ldots(x-n) + x = \frac{2013(-1)^n}{(n+1)!}(x-1)(x-2)\ldots(x-n) + x. \tag{4}$$

   By (3), condition (ii) holds if and only if $n + 1$ is a divisor of 2013. Thus, any polynomial $P(x)$ of degree n satisfying all three conditions (i)–(iii) must be of the form (4) with $n + 1$ a divisor of 2013.

   Conversely, it is easy to see that any polynomial of the form (4), where $n + 1$ divides 2013, is a polynomial of degree n satisfying the conditions (i)–(iii). Therefore the numbers n sought in the problem are exactly the positive integers of the form $n = d - 1$, where d is a divisor of 2013.

3. Given positive integers n and m with $n \geq 2m$, let $f(n, m)$ be the number of binary sequences of length n (i.e., strings $a_1 a_2 \ldots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. For example, the sequence $100\boxed{01}111\boxed{01}00\boxed{01}0$ contains this block 3 times.

   Find, with proof, a simple formula for $f(n, m)$.

   Solution. Every sequence of the required form can be written as $B_1 C_1 \boxed{01} B_2 C_2 01 \ldots \boxed{01} B_{m+1} C_{m+1}$, where each B_i is a block of 1's and each C_i a block of 0's, with empty blocks being allowed, and the sum of the lengths of the blocks B_i and C_i is $n - 2m$. Moreover, the sequence is uniquely determined by the $(2m+2)$-tuple (1) $(b_1, c_1, b_2, c_2, \ldots, b_{m+1}, c_{m+1})$ where b_i and c_i denote the number of elements in the blocks B_i and C_i, respectively. (For example, the sequence given in the problem can be represented as $\boxed{1}\boxed{00}\mathbf{01}\boxed{111}\boxed{}\mathbf{01}\boxed{}\boxed{00}\mathbf{01}\boxed{}\boxed{0}$ and thus is uniquely described by the 8-tuple $(1, 2, 3, 0, 0, 2, 0, 1)$.) Conversely, any tuple of the form (1) with nonnegative integers b_i and c_i satisfying $\sum_{i=1}^{m+1}(b_i + c_i) = n - 2m$ determines a sequence of the required type. Hence the number of such sequences is equal to the number of ways one can write $n - 2m$ as a sum of $2m + 2$ nonnegative integers, with order taken into account. The latter problem is equivalent to counting the number of ways of choosing $2n-m$ donuts from $2m+2$ varieties, a well-known combinatorial problem whose answer is given by the binomial coefficient $\binom{a}{b}$ with $a=(n-2m)+(2m+2)-1=n+1$ and $b=(2m+2)-1=2m+1$. Hence $f(n,m)=\binom{n+1}{2m+1}$.

4. Let $x_0=0$, $x_1=1$, and

$$x_{n+1}=\frac{1}{n+1}x_n+\left(1-\frac{1}{n+1}\right)x_{n-1}\quad (n\geq 1).$$

Show that the sequence $\{x_n\}$ converges as $n\to\infty$ and determine its limit.

Solution. The given recurrence can be written as

$$x_{n+1}(n+1)=x_n+nx_{n-1}\quad (n\geq 1).$$

Setting $d_n=x_{n+1}-x_n$ and simplifying, we deduce $d_n=(-n/(n+1))d_{n-1}$ for $n\geq 1$. Iterating this relation, we get

$$d_n=\frac{-n}{n+1}\cdot\frac{-(n-1)}{n}\cdots\frac{-1}{2}d_0=\frac{(-1)^n}{n+1},$$

since $d_0=x_1-x_0=1$. Hence

$$x_n=x_0+\sum_{i=0}^{n-1}d_i=\sum_{i=0}^{n-1}\frac{(-1)^i}{i+1}.$$

The last series is an alternating series with decreasing terms and thus converges. Its sum, $\sum_{i=0}^{\infty}(-1)^i/(i+1)$ equals $\ln 2$ by the Taylor series for $\ln(1+x)$.

5. [A4, Putnam 1998] Let $A_1=0$, $A_2=1$, and for $n>2$ define A_n as the number obtained by concatenating the numbers A_{n-1} and A_{n-2} (written in decimal). Thus, $A_3=A_2A_1=\boxed{1}\boxed{0}=10$, $A_4=A_3A_2=\boxed{10}\boxed{1}=101$, $A_5=A_4A_3=\boxed{101}\boxed{10}=10110$, and so on.

Determine, with proof, the set of n for which A_n is divisible by 11.

Solution. Let $|A_n|$ denote the number of digits of A_n. By definition, $|A_1|=|A_2|=1$ and for $n\geq 3$ the recurrence for A_n yields $|A_{n+1}|=|A_n|+|A_{n-1}|$. Hence $|A_n|$ satisfies the Fibonacci recurrence with the same initial conditions, therefore must be equal to the n-th Fibonacci number F_n. The recurrence therefore can be written as

$$A_n=10^{F_{n-2}}A_{n-1}+A_{n-2}.$$

Reducing modulo 11, we obtain

$$A_n\equiv(-1)^{F_{n-2}}A_{n-1}+A_{n-2}\bmod 11.$$

Since F_n is even if and only if n is divisible by 3 (a fact that is easy to prove by induction), this becomes

(1)
$$A_n\equiv\begin{cases}A_{n-1}+A_{n-2}\bmod 11 & \text{if } n\equiv 2\bmod 3,\\ -A_{n-1}+A_{n-2}\bmod 11 & \text{if } n\equiv 0,1\bmod 3,\end{cases}$$

Checking the first few cases directly, we see that A_n is congruent modulo 11 to $0,1,-1,2,1,1,0,1$ for $n=1,2,\ldots,8$, suggesting that $(*)$ $A_{n+6}\equiv A_n\bmod 11$ for all $n\in\mathbb{N}$. $(*)$ can be proved with a routine induction argument, using the congruences for $=1,2,\ldots,6$ as base case, and (1) for the induction step.

It follows that $A_n\equiv 0\bmod 11$ holds if and only if n is of the form $n=1+6k$ for some nonnegative integer k.

6. [B4, Putnam 1988] Let $a_1,a_2,a_3,\ldots$ be a sequence of positive real numbers, and let $A_n=\sqrt[n]{a_n}$. Prove that if the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{a_n}$ converges, then so does the series $\displaystyle\sum_{n=1}^{\infty}\frac{A_n}{a_n}$.

Solution. Suppose $\sum_{n=1}^{\infty}1/a_n$ converges. We split the set of positive integers $\mathbb{N}$ into sets $\mathbb{N}_1$ and $\mathbb{N}_2$ defined by

$$\mathbb{N}_1=\{n\in\mathbb{N}:a_n\leq 2^n\},\quad \mathbb{N}_2=\{n\in\mathbb{N}:a_n>2^n\}.$$

It suffices to show that for $i=1,2$,

(*)
$$\sum_{n\in\mathbb{N}_i}\frac{A_n}{a_n}<\infty.$$

For $n \in \mathbb{N}_1$ we have $A_n = a_n^{1/n} < (2^n)^{1/n} = 2$. Hence

$$\sum_{n \in \mathbb{N}_1} \frac{A_n}{a_n} < \sum_{n \in \mathbb{N}} \frac{2}{a_n} < \infty,$$

so $(*)$ holds for $i = 1$.
If $n \in \mathbb{N}_2$ and $n \geq 2$, then $A_n/a_n = a_n^{(1/n)-1} \leq a_n^{-1/2} < 2^{-n/2}$ since $a_n > 2^n$ for $n \in \mathbb{N}_2$. Thus,

$$\sum_{n \in \mathbb{N}_2} \frac{A_n}{a_n} < \frac{A_1}{a_1} + \sum_{n \geq 2} 2^{-n/2} < \infty,$$

so $(*)$ holds for $i = 2$ as well, and the proof is complete.