2012 UI MOCK PUTNAM EXAM

1. Prove that, given any power of 2 (such as 1024), there exist infinitely many powers of 2 whose decimal representation ends with the digits of the given power of 2.

2. Determine, with proof, all positive integers n for which there is a polynomial of degree n satisfying the following three conditions:

   (i) $P(k) = k$ for $k = 1, 2, \ldots, n$;

   (ii) $P(0)$ is an integer;

   (iii) $P(-1) = 2012$.

3. Given positive integers n and m with $n \geq 2m$, let $f(n, m)$ be the number of binary sequences of length n (i.e., strings $a_1 a_2 \ldots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. For example, the sequence $100\boxed{01}111\boxed{01}00\boxed{01}0$ contains this block 3 times.

   Find, with proof, a simple formula for $f(n, m)$.

4. Let $x_0 = 0$, $x_1 = 1$, and
$$x_{n+1} = \frac{1}{n+1} x_n + \left(1 - \frac{1}{n+1}\right) x_{n-1} \quad (n \geq 1).$$
   Show that the sequence $\{x_n\}$ converges as $n \to \infty$ and determine its limit.

5. [A4, Putnam 1998] Let $A_1 = 0$, $A_2 = 1$, and for $n > 2$ define A_n as the number obtained by concatenating the numbers A_{n-1} and A_{n-2} (written in decimal). Thus, $A_3 = A_2 A_1 = \boxed{1}\boxed{0} = 10$, $A_4 = A_3 A_2 = \boxed{10}\boxed{1} = 101$, $A_5 = A_4 A_3 = \boxed{101}\boxed{10} = 10110$, and so on.

   Determine, with proof, the set of n for which A_n is divisible by 11.

6. [B4, Putnam 1988] Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and let $A_n = \sqrt[n]{a_n}$. Prove that if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, then so does the series $\displaystyle\sum_{n=1}^{\infty} \frac{A_n}{a_n}$.

[Solutions at `http://www.math.uiuc.edu/contests.html`]