2011 U OF I MOCK PUTNAM CONTEST

Solutions

1. [Variation of A2, Putnam 1987] The sequence of digits

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 1\ 0\ 1\ 1\ 1\ 2\ 1\ 3\ \ldots$$

is obtained by writing out the natural numbers in order. Let $f(n)$ denote the position of the first digit of the number n in this sequence. Thus, for example, $f(1) = 1$, $f(2) = 2$, $f(10) = 10$ (since the integer $n = 10$ occupies positions 10 and 11 in this sequence), $f(11) = 12$ (since 11 occupies positions 12 and 13), $f(12) = 14$ (since 12 occupies positions 14 and 15, and so on.

Find, with proof, a simple explicit formula for $f(10^k)$, where k is an arbitrary positive integer.

Solution. The desired formula is $\boxed{f(10^k = k10^k - (10^k - 1)/9 + 1}$. To prove this, note that 10^k is the first integer with $k + 1$ digits. Thus, the position of its first digit is 1 plus the total number of positions occupied by the integers with at most k digits.

Now there are $10^i - 10^{i-1}$ integers with exactly i digits (namely, all integers n in the range $10^{i-1} \le n < 10^i$), so the total number of positions occupied by integers with i digits is $i(10^i - 10^{i-1})$. Adding these counts for $i = 1, 2, \ldots, k - 1$ gives the number of positions occupied by integers with at most k digits:

$$\begin{aligned}\sum_{i=1}^{k}(10^i - 10^{i-1})i &= \sum_{i=1}^{k} 10^i i - \sum_{j=0}^{k-1} 10^j (j + 1)\\ &= 10^k k - \sum_{j=0}^{k-1} 10^j\\ &= 10^k k - \frac{10^k - 1}{10 - 1}.\end{aligned}$$

Adding 1 to this count gives the above formula for $f(10^k)$.

2. Let $a_n = [(\sqrt{2} + 1)^n]$, where $[x]$ denotes the greatest integer $\le x$ (i.e., the floor function). Prove that a_n is even if and only if n is odd.

Solution. Let $b_n := (1 + \sqrt{2})^n + (1 - \sqrt{2})^n$. Expanding each of the two nth powers by the binomial theorem shows that

$$b_n = \sum_{k=0}^{n} \binom{n}{k} \left((\sqrt{2})^k + (-\sqrt{2})^k\right) = 2 \sum_{k \text{ even}} \binom{n}{k} 2^{k/2}.$$

Hence b_n is an even integer. Now $(1 - \sqrt{2})^n$ is a number of absolute value at most $\sqrt{2} - 1 < 1/2$ and is negative for n odd, and positive for n even. Since $(\sqrt{2} + 1)^n = b_n - (1 - \sqrt{2})^n$, by the definition of b_n, it follows that $[(\sqrt{2} + 1)^n]$ is equal to b_n when n is odd, and equal to $b_n - 1$ when n is even. Since b_n is always even, this proves the claim.

3. [A2, Putnam 2003] Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers. Prove that

$$(a_1 \ldots a_n)^{1/n} + (b_1 \ldots b_n)^{1/n} \le (a_1 + b_1)^{1/n} \ldots (a_n + b_n)^{1/n}.$$

Solution. Dividing by the right-hand side, the given inequality can be rewritten as

$$(x_1 \ldots x_n)^{1/n} + (y_1 \ldots y_n)^{1/n} \le 1, \tag{1}$$

where

$$x_i = \frac{a_i}{a_i + b_i}, \quad y_i = \frac{b_i}{a_i + b_i}.$$

Applying AGM to each term on the left of (1) gives

$$(x_1 \ldots x_n)^{1/n} + (y_1 \ldots y_n)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(x_i + y_i) = 1,$$

since $x_i + y_i = 1$ for each i.

4. Let $P_1, P_2, P_3, \dots$ be a sequence of points in 3-dimensional space satisfying (i) $|P_i| \geq 1$ for all i and (ii) $|P_iP_j| \geq 1$ for all i and j with $i \neq j$. (Here $|PQ|$ denotes the usual (Euclidean) distance between P and Q, and $|P|$ denotes the distance between P and the origin.)

(a) Prove that, if $\alpha > 3$, then the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{|P_i|^\alpha}$$

converges.

(b) Show that there exists a sequence of points P_i satisfying conditions (i) and (ii) above for which the above series diverges when $\alpha = 3$.

Solution. (a) First note that, without loss of generality, we may assume that the P_i all lie in the first octant. Subdivide this octant into boxes of side length 1/2 of the form

$$B(m,n,p) = \left[\frac{m}{2}, \frac{m+1}{2}\right] \times \left[\frac{n}{2}, \frac{n+1}{2}\right] \times \left[\frac{p}{2}, \frac{p+1}{2}\right],$$

where m, n, p are nonnegative integers. Since the largest distance between any two points in such a box is $\sqrt{3/4} < 1$, such a box can contain at most one of the points P_i. Moreover, since $|P_i| \geq 1$, the box $B(0,0,0)$ cannot contain a point P_i.

Now note that, if $P_i \in B(m,n,p)$, then

$$|P_i| \geq \left|\left(\frac{m}{2}, \frac{n}{2}, \frac{p}{2}\right)\right| = \frac{1}{2}\sqrt{m^2+n^2+p^2}.$$

Hence

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{|P_i|^\alpha} &\leq \sum_{\substack{m,n,p=0\\ \max(m,n,p)>0}}^{\infty} \frac{2^\alpha}{(m^2+n^2+p^2)^{\alpha/2}} \\
&\leq 3!2^\alpha \sum_{\substack{0\leq m\leq n\leq p\\ p\neq 0}}^{\infty} \frac{1}{(m^2+n^2+p^2)^{\alpha/2}} \quad \text{(using symmetry in } m,n,p) \\
&\leq 6\cdot 2^\alpha \sum_{\substack{0\leq m\leq n\leq p\\ p\neq 0}}^{\infty} \frac{1}{p^\alpha} \quad (\text{since } m^2+n^2+p^2 \geq p^2) \\
&\leq 6\cdot 2^\alpha \sum_{p=1}^{\infty} \frac{(p+1)^2}{p^\alpha} \quad (\text{since there are} \leq (p+1)^2 \text{ choices of } (m,n) \text{ with } m,n \leq p) \\
&\leq 6\cdot 2^\alpha \sum_{p=1}^{\infty} \frac{(2p)^2}{p^\alpha} \quad (\text{since } p+1 \leq 2p) \\
&= 24\cdot 2^\alpha \sum_{p=1}^{\infty} \frac{1}{p^{\alpha-2}},
\end{aligned}$$

and since $\alpha > 3$, the latter series converges. This proves part (a).

(b) If we let the P_i be the lattice points (m,n,p) with positive integral coordinates, then the conditions $|P_i| \geq 1$ and

$|P_iP_j| \geq 1$ are satisfied. On the other hand, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{|P_i|^3} &\geq \sum_{m,n,p=1}^{\infty} \frac{1}{(m^2+n^2+p^2)^{3/2}} \\
&\geq \sum_{k=1}^{\infty} \sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^k}^{2^{k+1}-1} \sum_{p=2^k}^{2^{k+1}-1} \frac{1}{(m^2+n^2+p^2)^{3/2}} \\
&\geq \sum_{k=1}^{\infty} \sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^k}^{2^{k+1}-1} \sum_{p=2^k}^{2^{k+1}-1} \frac{1}{100 \cdot 2^{3k}} \quad (\text{since } (m^2+n^2+p^2)^{3/2} \leq (3 \cdot 2^{2k+2})^{3/2} \leq 100 \cdot 2^{3k}) \\
&= \frac{1}{100} \sum_{k=1}^{\infty} (2^k)^3 \frac{1}{2^{3k}} \quad (\text{since there are } (2^k)^3 \text{ choices of } (m,n,p)) \\
&= \frac{1}{100} \sum_{k=1}^{\infty} 1 = \infty,
\end{aligned}$$

so the series $\sum_{i=1}^{\infty} 1/|P_i|^3$ diverges.

5. Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2+\cos(2\pi \ln n)}}$$

converges.

Solution. We claim that the series diverges (though only barely so). The proof is somewhat technical, but the underlying idea is easy to describe: Namely, focus on integers for which $(*)$ $\ln n \approx k + 1/2$, where k is a positive integer. In this case, $\cos(2\pi \ln n) \approx \cos(2\pi(k+1/2)) = -1$, so the exponent of n in the given series will be approximately 1. Thus, if one can show that $(*)$ holds for long enough intervals of n's, then the series behaves like a harmonic series over these stretches and therefore diverges.

To make this precise, define the intervals I_k, for $k = 1, 2, 3, \ldots$, by

$$I_k = \left[e^{k+1/2}, e^{k+1/2+1/(2k)}\right).$$

Note that these intervals are disjoint. We will show that the given sum restricted to the interval I_k is $\geq c/k$, where c is a positive constant. Hence the entire series is bounded from below by $\geq \sum_{k=1}^{\infty} c/k$, a divergent series.

To prove this, note first that

$$\begin{aligned}
n \in I_k &\iff e^{k+\frac{1}{2}} \leq n < e^{k+\frac{1}{2}+\frac{1}{2k}} \\
&\iff k + \frac{1}{2} \leq \ln n < k + \frac{1}{2} + \frac{1}{2k} \\
&\implies \cos\left(2\pi\left(k+\frac{1}{2}\right)\right) \leq \cos(2\pi \ln n) < \cos\left(2\pi\left(k+\frac{1}{2}+\frac{1}{2k}\right)\right) \\
&\iff \cos(\pi) \leq \cos(2\pi \ln n) < \cos\left(\pi\left(1+\frac{1}{k}\right)\right) \\
&\implies -1 \leq \cos(2\pi \ln n) < -1 + \frac{\pi}{k},
\end{aligned}$$

where the latter inequality follows from the bound

$$\cos(\pi(1+x)) \leq \cos(\pi) + x \max_{0 \leq t \leq x} |\pi \sin(\pi(1+t))| \leq -1 + \pi x.$$

Also, if $n \in I_k$, then

$$\ln n < k + \frac{1}{2} + \frac{1}{2k} \leq k + 1 \leq 2k,$$

$$\frac{1}{n^{1+\pi/k}} = \frac{e^{-\pi(\ln n)/k}}{n} \geq \frac{e^{-2\pi}}{n}.$$

It follows that

$$
\begin{aligned}
\sum_{n\in I_k} \frac{1}{n^{2+\cos(\pi \ln n)}} &> \sum_{n\in I_k} \frac{e^{-2\pi}}{n}. \\
&\geq e^{-2\pi}\left(\int_{I_k} \frac{1}{t}dt - \frac{1}{e^{k+1/2}}\right), \\
&= e^{-2\pi}\left(\ln(e^{k+1/2+1/k}) - \ln(e^{k+1/2}) - \frac{1}{e^{k+1/2}}\right), \\
&\geq e^{-2\pi}\left(\frac{1}{k} - e^{-k}\right).
\end{aligned}
$$

Summing the latter expression over all k gives a divergent series. Hence the given series diverges as well.

6. Let G_n denote the geometric mean of the n binomial coefficients $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$.

Prove that the limit $\lim_{n\to\infty} \sqrt[n]{G_n}$ exists, and find its value. (You may only use the definition of binomial coefficients and standard results from calculus, but not, for example, asymptotic formulas for binomial coefficients or factorials.)

Solution. We will show that $\lim_{n\to\infty} G_n^{1/n} = \sqrt{e}$. Using the definition of the binomial coefficients, we get

$$
\begin{aligned}
G_n^n = \prod_{k=1}^{n} \binom{n}{k} = \prod_{k=1}^{n} \frac{n!}{k!(n-k)!} &= \frac{n!^n}{1!^2 2!^2 \dots n!^2} \\
&= \frac{1^n \cdot 2^n \cdot \dots \cdot n^n}{1^{2n}2^{2n-2}3^{2n-4}\dots(n-1)^4 n^2}.
\end{aligned}
$$

Note that, for each $i \in \{1, 2, \dots, n\}$, the factor i occurs exactly n times in the numerator and $2(n-i+1)$ times in the denominator. Hence,

$$
\begin{aligned}
\ln G_n^n &= \sum_{i=1}^{n} (\ln i)(n - 2(n-i+1)) \\
&= \sum_{i=1}^{n} (\ln i)(2i - n) - 2\sum_{i=1}^{n} \ln i. \qquad (1)
\end{aligned}
$$

Also,

$$
\sum_{i=1}^{n} (\ln n)(2i - n) = (\ln n)\left(2\frac{n(n+1)}{2} - n^2\right) = n \ln n. \qquad (2)
$$

Subtracting (2) from (1) gives

$$
\ln G_n^n = \sum_{i=1}^{n} (\ln i - \ln n)(2i - n) - 2\sum_{i=1}^{n} \ln i + n \ln n.
$$

Dividing by n^2 we get

$$
\ln G_n^{1/n} = \frac{1}{n^2} \ln G_n^n = \frac{1}{n}\sum_{i=1}^{n} \left(\ln \frac{i}{n}\right)\left(2\frac{i}{n} - 1\right) + R(n), \qquad (3)
$$

where

$$
R(n) = \frac{\ln n}{n} - \frac{2}{n^2}\sum_{i=1}^{n} \ln i + \frac{\ln n}{n}.
$$

Since

$$
|R(n)| \leq \frac{\ln n}{n} + \frac{2}{n^2}\sum_{i=1}^{n} \ln n \leq \frac{3 \ln n}{n},
$$

we have $\lim_{n\to\infty} R(n) = 0$, so the term $R(n)$ has no effect on the limit of $\ln G_n^{1/n}$, and we can therefore ignore it. The remaining term on the right of (3) can be interpreted as a Riemann sum for the integral $\int_0^1 (\ln x)(2x-1)dx$, and evaluating this integral gives the desired limit:

$$\begin{aligned}
\lim_{n\to\infty} \ln G_n^{1/n} &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\ln \frac{i}{n}\right)\left(2\frac{i}{n}-1\right) \\
&= \int_0^1 (\ln x)(2x-1)dx \\
&= (x^2-x)\ln x\Big|_0^1 - \int_0^1 (x^2-x)\frac{1}{x}dx \\
&= -\int_0^1 (x-1)dx = \frac{1}{2}.
\end{aligned}$$

Exponentiating, we get $\lim_{n\to\infty} G_n^{1/n} = e^{1/2}$, as claimed.