1. [Variation of A2, Putnam 1987] The sequence of digits 1 2 3 4 5 6 7 8 9 1 0 1 1 1 2 1 3 \ldots

is obtained by writing out the natural numbers in order. Let \( f(n) \) denote the position of the first digit of the number \( n \) in this sequence. Thus, for example, \( f(1) = 1, f(2) = 2, f(10) = 10 \) (since the integer \( n = 10 \) occupies positions 10 and 11 in this sequence), \( f(11) = 12 \) (since 11 occupies positions 12 and 13), \( f(12) = 14 \) (since 12 occupies positions 14 and 15, and so on).

Find, with proof, a simple explicit formula for \( f(10^k) \), where \( k \) is an arbitrary positive integer.

**Solution.** The desired formula is \( f(10^k) = k10^k - (10^k - 1)/9 + 1 \). To prove this, note that \( 10^k \) is the first integer with \( k + 1 \) digits. Thus, the position of its first digit is 1 plus the total number of positions occupied by the integers with at most \( k \) digits.

Now there are \( 10^i - 10^{i-1} \) integers with exactly \( i \) digits (namely, all integers \( n \) in the range \( 10^{i-1} \leq n < 10^i \)), so the total number of positions occupied by integers with \( i \) digits is \( i(10^i - 10^{i-1}) \). Adding these counts for \( i = 1, 2, \ldots, k - 1 \) gives the number of positions occupied by integers with at most \( k \) digits:

\[
\sum_{i=1}^{k} (10^i - 10^{i-1}) i = \sum_{i=1}^{k} 10^i i - \sum_{j=0}^{k-1} 10^j (j + 1)
= 10^k k - \sum_{j=0}^{k-1} 10^j
= 10^k k - \frac{10^k - 1}{10 - 1}.
\]

Adding 1 to this count gives the above formula for \( f(10^k) \).

2. Let \( a_n = \lfloor (\sqrt{2} + 1)^n \rfloor \), where \( \lfloor x \rfloor \) denotes the greatest integer \( \leq x \) (i.e., the floor function). Prove that \( a_n \) is even if and only if \( n \) is odd.

**Solution.** Let \( b_n := (1 + \sqrt{2})^n + (1 - \sqrt{2})^n \). Expanding each of the two \( n \)th powers by the binomial theorem shows that

\[
b_n = \sum_{k=0}^{n} \binom{n}{k} \left((\sqrt{2})^k + (-\sqrt{2})^k\right) = 2 \sum_{k \text{ even}} \binom{n}{k} 2^{k/2}.
\]

Hence \( b_n \) is an even integer. Now \( (1 - \sqrt{2})^n \) is a number of absolute value at most \( \sqrt{2} - 1 < 1/2 \) and is negative for \( n \) odd, and positive for \( n \) even. Since \( (\sqrt{2} + 1)^n = b_n - (1 - \sqrt{2})^n \), by the definition of \( b_n \), it follows that \( \lfloor (\sqrt{2} + 1)^n \rfloor \) is equal to \( b_n \) when \( n \) is odd, and equal to \( b_n - 1 \) when \( n \) is even. Since \( b_n \) is always even, this proves the claim.

3. [A2, Putnam 2003] Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be positive real numbers. Prove that

\[
(a_1 \ldots a_n)^{1/n} + (b_1 \ldots b_n)^{1/n} \leq (a_1 + b_1)^{1/n} \ldots (a_n + b_n)^{1/n}.
\]

**Solution.** Dividing by the right-hand side, the given inequality can be rewritten as

\[
(x_1 \ldots x_n)^{1/n} + (y_1 \ldots y_n)^{1/n} \leq 1,
\]

where

\[
x_i = \frac{a_i}{a_i + b_i}, \quad y_i = \frac{b_i}{a_i + b_i}.
\]

Applying AGM to each term on the left of (1) gives

\[
(x_1 \ldots x_n)^{1/n} + (y_1 \ldots y_n)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i + \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{n} \sum_{i=1}^{n} (x_i + y_i) = 1,
\]

since \( x_i + y_i = 1 \) for each \( i \).
4. Let $P_1, P_2, P_3, \ldots$ be a sequence of points in 3-dimensional space satisfying (i) $|P_i| \geq 1$ for all $i$ and (ii) $|P_iP_j| \geq 1$ for all $i$ and $j$ with $i \neq j$. (Here $|PQ|$ denotes the usual (Euclidean) distance between $P$ and $Q$, and $|P|$ denotes the distance between $P$ and the origin.)

(a) Prove that, if $\alpha > 3$, then the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{|P_i|^\alpha}$$

converges.

(b) Show that there exists a sequence of points $P_i$ satisfying conditions (i) and (ii) above for which the above series diverges when $\alpha = 3$.

Solution. (a) First note that, without loss of generality, we may assume that the $P_i$ all lie in the first octant. Subdivide this octant into boxes of side length $1/2$ of the form

$$B(m,n,p) = \left[ \frac{m}{2}, \frac{m+1}{2} \right] \times \left[ \frac{n}{2}, \frac{n+1}{2} \right] \times \left[ \frac{p}{2}, \frac{p+1}{2} \right],$$

where $m, n, p$ are nonnegative integers. Since the largest distance between any two points in such a box is $\sqrt{3/4} < 1$, such a box can contain at most one of the points $P_i$. Moreover, since $|P_1| \geq 1$, the box $B(0,0,0)$ cannot contain a point $P_i$.

Now note that, if $P_i \in B(m,n,p)$, then

$$|P_i| \geq \left| \left( \frac{m}{2}, \frac{n}{2}, \frac{p}{2} \right) \right| = \frac{1}{2} \sqrt{m^2 + n^2 + p^2}.$$

Hence

$$\sum_{i=1}^{\infty} \frac{1}{|P_i|^\alpha} \leq \sum_{\substack{m,n,p=0 \\text{max}(m,n,p) > 0}}^{\infty} \frac{2^\alpha}{(m^2 + n^2 + p^2)^{\alpha/2}}$$

$$\leq 3! \cdot 2^\alpha \sum_{\substack{0 \leq m \leq n \leq p \\text{p \neq 0}}} \frac{1}{(m^2 + n^2 + p^2)^{\alpha/2}}$$ (using symmetry in $m, n, p$)

$$\leq 6 \cdot 2^\alpha \sum_{\substack{0 \leq m \leq n \leq p \\text{p \neq 0}}} \frac{1}{p^\alpha}$$ (since $m^2 + n^2 + p^2 \geq p^2$)

$$\leq 6 \cdot 2^\alpha \sum_{p=1}^{\infty} \frac{(p+1)^2}{p^\alpha}$$ (since there are $\leq (p+1)^2$ choices of $(m, n)$ with $m, n \leq p$)

$$\leq 6 \cdot 2^\alpha \sum_{p=1}^{\infty} \frac{(2p)^2}{p^\alpha}$$ (since $p + 1 \leq 2p$)

$$= 24 \cdot 2^\alpha \sum_{p=1}^{\infty} \frac{1}{p^{\alpha-2}},$$

and since $\alpha > 3$, the latter series converges. This proves part (a).

(b) If we let the $P_i$ be the lattice points $(m, n, p)$ with positive integral coordinates, then the conditions $|P_i| \geq 1$ and
\(|P_iP_j| \geq 1\) are satisfied. On the other hand, we have

\[
\sum_{i=1}^{\infty} \frac{1}{|P_i|^3} \geq \sum_{m,n,p=1}^{\infty} \frac{1}{(m^2 + n^2 + p^2)^{3/2}}
\]

\[
\geq \sum_{k=1}^{\infty} \sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^k}^{2^{k+1}-1} \sum_{p=2^k}^{2^{k+1}-1} \frac{1}{(m^2 + n^2 + p^2)^{3/2}}
\]

\[
\geq \sum_{k=1}^{\infty} \sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^k}^{2^{k+1}-1} \sum_{p=2^k}^{2^{k+1}-1} \frac{1}{100 \cdot 2^{3k}} \quad \text{(since \(m^2 + n^2 + p^2)^{3/2} \leq (3 \cdot 2^{2k+2})^{3/2} \leq 100 \cdot 2^{3k}\))
\]

\[
= \frac{1}{100} \sum_{k=1}^{\infty} (2^k)^3 \frac{1}{2^{3k}}
\]

\[
= \frac{1}{100} \sum_{k=1}^{\infty} 1 = \infty,
\]

so the series \(\sum_{i=1}^{\infty} 1/|P_i|^3\) diverges.

5. Determine, with proof, whether the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^{2+\cos(2\pi \ln n)}},
\]

converges.

**Solution.** We claim that the series diverges (though only barely so). The proof is somewhat technical, but the underlying idea is easy to describe: Namely, focus on integers for which \((*) \ln n \approx k + 1/2\), where \(k\) is a positive integer.

In this case, \(\cos(2\pi \ln n) \approx \cos(2\pi(k + 1/2)) = -1\), so the exponent of \(n\) in the given series will be approximately 1. Thus, if one can show that \((*)\) holds for long enough intervals of \(n\)'s, then the series behaves like a harmonic series over these stretches and therefore diverges.

To make this precise, define the intervals \(I_k\), for \(k = 1, 2, 3, \ldots\), by

\[
I_k = \left[ e^{k+1/2}, e^{k+1/2}+1/(2k) \right].
\]

Note that these intervals are disjoint. We will show that the given sum restricted to the interval \(I_k\) is \(\geq c/k\), where \(c\) is a positive constant. Hence the entire series is bounded from below by \(\geq \sum_{k=1}^{\infty} c/k\), a divergent series.

To prove this, note first that

\[
n \in I_k \iff e^{k+1/2} \leq n < e^{k+1/2}+\frac{1}{2k}
\]

\[
\iff k + \frac{1}{2} \leq \ln n < k + \frac{1}{2} + \frac{1}{2k}
\]

\[
\iff \cos \left( 2\pi \left(k + \frac{1}{2}\right) \right) \leq \cos(2\pi \ln n) < \cos \left( 2\pi \left(k + \frac{1}{2} + \frac{1}{2k}\right) \right)
\]

\[
\iff \cos(\pi) \leq \cos(2\pi \ln n) < \cos \left( \pi \left(1 + \frac{1}{k}\right) \right)
\]

\[
\iff -1 \leq \cos(2\pi \ln n) < -1 + \frac{\pi}{k},
\]

where the latter inequality follows from the bound

\[
\cos(\pi(1 + x)) \leq \cos(\pi) + x \max_{0 \leq t \leq x} |\pi \sin(\pi(1 + t))| \leq -1 + \pi x.
\]

Also, if \(n \in I_k\), then

\[
\ln n < k + \frac{1}{2} + \frac{1}{2k} \leq k + 1 \leq 2k,
\]

\[
\frac{1}{n^{1+\pi/k}} = \frac{e^{-\pi \ln n/k}}{n} \geq \frac{e^{-2\pi}}{n}.
\]
It follows that
\[
\sum_{n \in I_k} \frac{1}{n^{2 + \cos(\pi n)}} > \sum_{n \in I_k} \frac{e^{-2\pi}}{n}.
\]
\[
\geq e^{-2\pi} \left( \int_{I_k} \frac{1}{t} dt - \frac{1}{e^{k+1/2}} \right),
\]
\[
= e^{-2\pi} \left( \ln(e^{k+1/2 + 1/k}) - \ln(e^{k+1/2}) - \frac{1}{e^{k+1/2}} \right),
\]
\[
\geq e^{-2\pi} \left( \frac{1}{k} - e^{-k} \right).
\]

Summing the latter expression over all \(k\) gives a divergent series. Hence the given series diverges as well.

6. Let \(G_n\) denote the geometric mean of the \(n\) binomial coefficients \(\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\).

Prove that the limit \(\lim_{n \to \infty} \sqrt[n]{G_n}\) exists, and find its value. (You may only use the definition of binomial coefficients and standard results from calculus, but not, for example, asymptotic formulas for binomial coefficients or factorials.)

**Solution.** We will show that \(\lim_{n \to \infty} \sqrt[n]{G_n} = \sqrt{e}\). Using the definition of the binomial coefficients, we get

\[
G_n = \prod_{k=1}^{n} \frac{n!}{k!(n-k)!} = \frac{n^n}{1^{1!}2^{2!}\ldots n^{n!}} = \frac{1^n 2^n \ldots n^n}{1^{2^n}2^{3^n-4\ldots(n-1)^4n^2}}.
\]

Note that, for each \(i \in \{1, 2, \ldots, n\}\), the factor \(i\) occurs exactly \(n\) times in the numerator and \(2(n-i+1)\) times in the denominator. Hence,

\[
\ln G_n = \sum_{i=1}^{n} (\ln i)(n - 2(n - i + 1))
\]
\[
= \sum_{i=1}^{n} (\ln i)(2i - n) - 2 \sum_{i=1}^{n} \ln i.
\]

(1)

Also,

\[
\sum_{i=1}^{n} (\ln n)(2i - n) = (\ln n) \left( \frac{n(n+1)}{2} - n^2 \right) = n \ln n.
\]

(2)

Subtracting (2) from (1) gives

\[
\ln G_n = \sum_{i=1}^{n} (\ln i - \ln n)(2i - n) - 2 \sum_{i=1}^{n} \ln i + n \ln n.
\]

Dividing by \(n^2\) we get

\[
\ln G_n^{1/n} = \frac{1}{n^2} \ln G_n = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\ln i}{n} \right) \left( 2 \frac{i}{n} - 1 \right) + R(n),
\]

(3)

where

\[
R(n) = \frac{\ln n}{n} - 2 \frac{1}{n^2} \sum_{i=1}^{n} \ln i + \frac{\ln n}{n}.
\]

Since

\[
|R(n)| \leq \frac{\ln n}{n} + \frac{2}{n^2} \sum_{i=1}^{n} \ln n \leq \frac{3 \ln n}{n},
\]

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we have $\lim_{n \to \infty} R(n) = 0$, so the term $R(n)$ has no effect on the limit of $\ln G_{n}^{1/n}$, and we can therefore ignore it. The remaining term on the right of (3) can be interpreted as a Riemann sum for the integral $\int_{0}^{1} (\ln x)(2x - 1)\,dx$, and evaluating this integral gives the desired limit:

$$\lim_{n \to \infty} \ln G_{n}^{1/n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \ln \frac{i}{n} \right) \left( 2 \frac{i}{n} - 1 \right)$$

$$= \int_{0}^{1} (\ln x)(2x - 1)\,dx$$

$$= (x^2 - x) \ln x \bigg|_{0}^{1} - \int_{0}^{1} (x^2 - x) \frac{1}{x} \,dx$$

$$= -\int_{0}^{1} (x - 1)\,dx = \frac{1}{2}.$$ 

Exponentiating, we get $\lim_{n \to \infty} G_{n}^{1/n} = e^{1/2}$, as claimed.