2011 U OF I MOCK PUTNAM CONTEST

1. [Variation of A2, Putnam 1987] The sequence of digits

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 1\ 0\ 1\ 1\ 1\ 2\ 1\ 3\ \ldots$$

   is obtained by writing out the natural numbers in order. Let $f(n)$ denote the position of the first digit of the number n in this sequence. Thus, for example, $f(1) = 1$, $f(2) = 2$, $f(10) = 10$ (since the integer $n = 10$ occupies positions 10 and 11 in this sequence), $f(11) = 12$ (since 11 occupies positions 12 and 13), $f(12) = 14$ (since 12 occupies positions 14 and 15, and so on.

   Find, with proof, a simple explicit formula for $f(10^k)$, where k is an arbitrary positive integer.

2. Let $a_n = [(\sqrt{2}+1)^n]$, where $[x]$ denotes the greatest integer $\le x$ (i.e., the floor function). Prove that a_n is even if and only if n is odd.

3. [A2, Putnam 2003] Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers. Prove that

$$(a_1 \ldots a_n)^{1/n} + (b_1 \ldots b_n)^{1/n} \le (a_1 + b_1)^{1/n} \ldots (a_n + b_n)^{1/n}.$$

4. Let $P_1, P_2, P_3, \ldots$ be a sequence of points in 3-dimensional space satisfying (i) $|P_i| \ge 1$ for all i and (ii) $|P_iP_j| \ge 1$ for all i and j with $i \ne j$. (Here $|PQ|$ denotes the usual (Euclidean) distance between P and Q, and $|P|$ denotes the distance between P and the origin.)

   (a) Prove that, if $\alpha > 3$, then the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{|P_i|^{\alpha}}$$

   converges.

   (b) Show that there exists a sequence of points P_i satisfying conditions (i) and (ii) above for which the above series diverges when $\alpha = 3$.

5. Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2+\cos(2\pi \ln n)}}$$

   converges.

6. Let G_n denote the geometric mean of the n binomial coefficients $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$.

   Prove that the limit $\lim_{n\to\infty} \sqrt[n]{G_n}$ exists, and find its value. (You may only use the definition of binomial coefficients and standard results from calculus, but not, for example, asymptotic formulas for binomial coefficients or factorials.)

[Solutions at `http://www.math.uiuc.edu/contests.html`]