2010 U OF I MOCK PUTNAM EXAM

1. (a) Given a set with n elements (where n is a positive integer), prove that exactly 2^{n-1} of its subsets have an odd number of elements.

   (b) Determine, with proof, the number of 8 by 8 matrices in which each entry is 0 or 1 and each row and each column contains an odd number of 1's.

2. A sheet of paper contains the numbers $101, 102, \ldots, 200$. Suppose you play the following game on this list of numbers. At each stage, you pick two of the numbers on the list, say a and b, cross these out, and replace them by the single number $ab + a + b$. You keep doing this until only a single number is left (which happens after 99 such moves). Determine, with proof, what this last number is.

3. Among all powers of 2, what percentage begin with the digit 1 in their decimal representation? More precisely, if $f(n)$ denotes the number of integers among the first n powers of 2 (i.e., $2^1, 2^2, \ldots, 2^n$) whose decimal representation begins with the digit 1, show that the limit $\lim_{n\to\infty} f(n)/n$ exists and compute its value.

4. Given a positive integer d, define a *lattice traversal of step size* d to be an infinite polygonal path $P_0P_1P_2\ldots$ in the plane satisfying the following conditions:

   (i) The distance between any two consecutive points P_i and P_{i+1} on the path is d.

   (ii) Each point P_i on the path is a lattice point (i.e., has integer coordinates).

   (iii) Each lattice point in the plane occurs at least once as a point P_i on the path.

   Determine, with proof, for which integers $d \in \{2, 3, \ldots, 10\}$ there exists a lattice traversal of step size d.

5. Let $1 \le a_1 < a_2 < a_3 \ldots$ be a sequence of positive integers, such that $a_k/k \to \infty$ as $k \to \infty$, and let $A(n)$ denote the number of terms in this sequence that are $\le n$. Prove that there exist infinitely many positive integers n that are divisible by $A(n)$.

6. Find, with proof, the precise set of real numbers α, such that any sequence x_n, $n = 1, 2, 3, \ldots$, of real numbers satisfying

   (1) $$\lim_{n\to\infty} (x_n - x_{n-2}) = 0.$$

   also satisfies

   (2) $$\lim_{n\to\infty} \frac{x_n}{n^\alpha} = 0.$$

[Solutions at `http://www.math.uiuc.edu/contests.html`]