1. (a) Given a set with \( n \) elements (where \( n \) is a positive integer), prove that exactly \( 2^{n-1} \) of its subsets have an odd number of elements.

(b) Determine, with proof, the number of 8 by 8 matrices in which each entry is 0 or 1 and each row and each column contains an odd number of 1’s.

2. A sheet of paper contains the numbers 101, 102, …, 200. Suppose you play the following game on this list of numbers. At each stage, you pick two of the numbers on the list, say \( a \) and \( b \), cross these out, and replace them by the single number \( ab + a + b \). You keep doing this until only a single number is left (which happens after 99 such moves). Determine, with proof, what this last number is.

3. Among all powers of 2, what percentage begin with the digit 1 in their decimal representation? More precisely, if \( f(n) \) denotes the number of integers among the first \( n \) powers of 2 (i.e., \( 2^1, 2^2, \ldots, 2^n \)) whose decimal representation begins with the digit 1, show that the limit \( \lim_{n \to \infty} \frac{f(n)}{n} \) exists and compute its value.

4. Given a positive integer \( d \), define a lattice traversal of step size \( d \) to be an infinite polygonal path \( P_0P_1P_2 \ldots \) in the plane satisfying the following conditions:

   (i) The distance between any two consecutive points \( P_i \) and \( P_{i+1} \) on the path is \( d \).

   (ii) Each point \( P_i \) on the path is a lattice point (i.e., has integer coordinates).

   (iii) Each lattice point in the plane occurs at least once as a point \( P_i \) on the path.

Determine, with proof, for which integers \( d \in \{2, 3, \ldots, 10\} \) there exists a lattice traversal of step size \( d \).

5. Let \( 1 \leq a_1 < a_2 < a_3 \ldots \) be a sequence of positive integers, such that \( a_k/k \to \infty \) as \( k \to \infty \), and let \( A(n) \) denote the number of terms in this sequence that are \( \leq n \). Prove that there exist infinitely many positive integers \( n \) that are divisible by \( A(n) \).

6. Find, with proof, the precise set of real numbers \( \alpha \), such that any sequence \( x_n, n = 1, 2, 3, \ldots \), of real numbers satisfying

   (1) \( \lim_{n \to \infty} (x_n - x_{n-2}) = 0 \).

   also satisfies

   (2) \( \lim_{n \to \infty} \frac{x_n}{n^\alpha} = 0 \).

   [Solutions at http://www.math.uiuc.edu/contests.html]