U OF I MOCK PUTNAM EXAM
SEPT. 29, 2009
Solutions

1. Suppose $P(x)$ is a polynomial with integer coefficients such that none of the values $P(1), \ldots, P(2009)$ is divisible by 2009. Prove that $P(n) \neq 0$ for all integers n.

   Solution. We use the fact that, for any integers a, b and $m \geq 1$, $a \equiv b \mod m$ implies $P(a) \equiv P(b) \mod m$. This follows from the general properties of congruences. Given an integer n, let $r \in \{1, 2, \ldots, 2009\}$ be such that $n \equiv r \mod 2009$. Then, by the above fact and the given hypothesis, $P(n) \equiv P(r) \not\equiv 0 \mod 2009$, and hence $P(n) \neq 0$.

2. Find a function $f(x)$ that satisfies, for all $x \geq 0$,

$$f(x) = \sqrt{\int_0^x (f(t)^2 + f'(t)^2)dt + 2009}.$$

   Solution. [Problem B1, Putnam 1990] Square the given equation and differentiate to get

$$2f(x)f'(x) = f(x)^2 + f'(x)^2,$$

   or

$$(f(x) - f'(x))^2 = 0,$$

   which is equivalent to $f'(x) = f(x)$. The latter equation has solution $f(x) = Ce^x$, where C is an arbitrary constant. Substituting this into the original equation, we get

$$Ce^x = \sqrt{C^2 \int_0^x 2(Ce^t)^2 dt + 2009} = \sqrt{C^2 e^{2x} - C^2 + 2009},$$

   and choosing $C = \sqrt{2009}$, we see that this equation is satisfied. Thus, $f(x) = \sqrt{2009}e^x$ is the desired solution.

3. Let $\mathcal{A} = (a_1, a_2, a_3, \ldots)$ be a permutation of the positive integers. (In other words, a_k is a positive integer for each k, and for each positive integer n, there exists exactly one k such that $a_k = n$.) Prove that $\mathcal{A}$ contains a triple (a_i, a_j, a_k) with $i < j < k$ and $a_j - a_i = a_k - a_j = d > 0$.

   Solution. Let j be the least integer such that $a_j > a_1$, and let k be the unique index such that $a_k = 2a_j - a_1$. Since $2a_j - a_1 > a_j > a_1$, it follows from the definition of the index j that $k > j$. The triple (a_1, a_j, a_k) has the required properties.

4. Let $b_1, \dots, b_n$ be integers greater than 1, and let $B = b_1 \dots b_n$ denote their product. Let d_i be the number of digits of B expressed in base b_i. For example, if $b_1 = 10$, $b_2 = 3$, $b_3 = 2$, then $B = 10 \cdot 3 \cdot 2 = (60)_{10} = (2020)_3 = (111100)_2$, so $d_1 = 2$, $d_2 = 4$, $d_3 = 6$. Show that
$$d_1 + \cdots + d_n > n^2.$$

Solution. First note that a positive integer m expressed in a base $b \geq 2$ has d digits if and only if $b^{d-1} \leq m < b^d$, or equivalently, $d - 1 \leq \ln m / \ln b < d$.

Now, set $\beta_i = \ln b_i$ and note that
$$\ln B = \ln(b_1 \dots b_n) = \sum_{j=1}^{n} \ln b_j = \sum_{i=j}^{n} \beta_j.$$

By the above inequality,
$$d_i > \frac{\ln B}{\ln b_i} = \frac{1}{\beta_i} \sum_{j=1}^{n} \beta_j.$$

Summing over $i = 1, 2, \dots, n$ and using the arithmetic-geometric mean inequality twice, we get
$$\begin{aligned} \sum_{i=1}^{n} d_i &> \left(\sum_{i=1}^{n} \frac{1}{\beta_i} \right) \left(\sum_{j=1}^{n} \beta_j \right) \\ &\geq n \left(\frac{1}{\beta_1} \cdots \frac{1}{\beta_n} \right)^{1/n} \left(\sum_{j=1}^{n} \beta_j \right) \\ &\geq n \left(\frac{1}{n} \sum_{i=1}^{n} \beta_i \right)^{-1} \left(\sum_{j=1}^{n} \beta_j \right) = n^2. \end{aligned}$$

Note: Instead of two applications of the AGM inequality, we could have used the arithmetic-harmonic mean inequality:
$$\frac{1}{n} \left(\sum_{i=1}^{n} \beta_i \right) \geq \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\beta_i} \right)^{-1}.$$

A third method is to use Cauchy's inequality in the form
$$n^2 = \left(\sum_{i=1}^{n} \beta_i^{-1/2} \beta_i^{1/2} \right)^2 \leq \left(\sum_{i=1}^{n} \frac{1}{\beta_i} \right) \left(\sum_{i=1}^{n} \beta_i \right).$$

5. Show that any positive integer containing exactly 2009 digits (in decimal), none of whose digits is zero, is either divisible by 2009, or can be changed to an integer that is divisible by 2009 by replacing some, but not all, of its digits by 0.

Solution. Let $n = a_1a_2 \dots a_{2009}$ be the given integer, let $n_0 = 0$, and for $k = 1, 2, \dots, 2009$ let n_k denote the number obtained by replacing all but the first k digits of n by the digit 0, i.e., $n_k = a_1a_2 \dots a_k00 \dots 0$. By the pigeon hole principle, two of the 2010 numbers $n_0, n_1, \dots, n_{2009}$, say n_{k_1} and n_{k_2} with $k_2 > k_1$, must be congruent modulo 2009. It follows that the difference $n_{k_2} - n_{k_1}$ is divisible by 2009. Now $n_{k_2} - n_{k_1}$ is the number obtained from n by replacing the first k_1 and the last $(2009 - k_2)$ digits by 0. Since $k_2 > k_1$, we have $k_1 + (2009 - k_2) < 2009$, so not all of the 2009 digits are replaced by zero. Thus, the number $n_{k_2} - n_{k_1}$ has the required properties.

6. Let $\mathcal{A}$ be an infinite set of positive integers, and let $A(n)$ denote the number of elements of $\mathcal{A}$ that are $\leq n$. Suppose that the series

$$\sum_{a \in \mathcal{A}} \frac{1}{a}$$

converges. Show that $\lim_{n\to\infty} A(n)/n = 0$.

Solution. [Problem B5, Putnam 1969] We argue by contradiction. Suppose $A(n)/n$ does not tend to 0. Then there exists $\epsilon > 0$ and an infinite sequence $n_1 < n_2 < \cdots$ of positive integers such that $A(n_k)/n_k \geq \epsilon$ for all k. Let N be a fixed positive integer. Since $n_k \to \infty$ as $k \to \infty$, we have $n_k > N$ for all large enough k. For such k we have

$$\sum_{\substack{a \in \mathcal{A} \\ N < a \leq n_k}} \frac{1}{a} \geq \sum_{\substack{a \in A \\ N < a \leq n_k}} \frac{1}{n_k} \geq \frac{A(n_k) - A(N)}{n_k} \geq \epsilon - \frac{A(N)}{n_k}.$$

Letting $k \to \infty$, we conclude

$$\sum_{a \in \mathcal{A}, a > N} \frac{1}{a} \geq \epsilon.$$

Since N was arbitrary, the series $\sum_{a\in\mathcal{A}} 1/a$ cannot be convergent.