

U OF I MOCK PUTNAM EXAM  
SEPT. 29, 2008

1. A sequence  $a_0, a_1, a_2 \dots$  of real numbers is defined recursively by

$$a_0 = 1, \quad a_{n+1} = \frac{a_n}{1 + na_n} \quad (n = 0, 1, 2, \dots).$$

Find a general formula for  $a_n$ .

2. Consider a  $7 \times 7$  checkerboard with the squares at the four corners removed (so that the remaining board has 45 squares). Is it possible to cover this board with  $1 \times 3$  tiles so that no two tiles overlap? Explain!
3. Let  $f$  be a function on  $[0, 2\pi]$  with continuous first and second derivatives and such that  $f''(x) > 0$  for  $0 < x < 2\pi$ . Show that the integral  $\int_0^{2\pi} f(x) \cos x \, dx$  is positive.
4. Given a nonnegative integer  $b$ , call a nonnegative integer  $a \leq b$  a *subordinate* of  $b$  if each decimal digit of  $a$  is at most equal to the decimal digit of  $b$  in the same position (counted from the right). For example, 1329 and 316 are subordinates of 1729, but 1338 is not since the second-last digit of 1338 is greater than the corresponding digit in 1729. Let  $f(b)$  denote the number of subordinates of  $b$ . For example,  $f(13) = 8$ , since 13 has exactly 8 subordinates: 13, 12, 11, 10, 3, 2, 1, 0. Find a simple formula for the sum

$$S(n) = \sum_{0 \leq b < 10^n} f(b).$$

5. Let  $a_1, a_2, \dots, a_{65}$  be positive integers, none of which has a prime factor greater than 13. Prove that, for some  $i, j$  with  $i \neq j$ , the product  $a_i a_j$  is a perfect square.
6. Let  $n$  be an *even* positive integer, and let  $S_n$  denote the set of all permutations of  $\{1, 2, \dots, n\}$ . Given two permutations  $\sigma_1, \sigma_2 \in S_n$ , define their distance  $d(\sigma_1, \sigma_2)$  by

$$d(\sigma_1, \sigma_2) = \sum_{k=1}^n |\sigma_1(k) - \sigma_2(k)|$$

Determine, with proof, the maximal distance between two permutations in  $S_n$ , i.e., determine the exact value of  $\max_{\sigma_1, \sigma_2 \in S_n} d(\sigma_1, \sigma_2)$ .

[Solutions at <http://www.math.uiuc.edu/contests.html>]